MATH 425
LECTURE 4
: Fundamental PDEs II.

We continue the derivation from physical principles of the prototypical PDEs. Last lecture we obtained the heat equation (also called the diffusion equation).

Now we will proceed with the transport and wave equation.

2| Derivation of the transport equation.

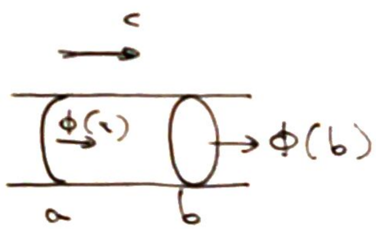
Consider some substance moving at constant velocity c along a 1-dimensional line (for example, a pollutant suspended in a liquid which moves along a pipe at constant velocity c).

Denote $\rho(x,t) \equiv$ (mass) density (of the pollutant)

PDE for ρ?

[Remark: We consider that the diffusion effects are negligible.]

• Mass conservation:



Mass in $[a,b]$ at $t+h$ = Mass in $[a,b]$ at t + Mass that has come in between t and $t+h$ − Mass " " out " " .

That is,

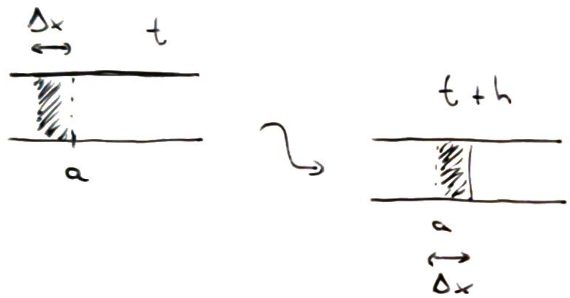
$$\int_a^b \rho(x,t+h)\,dx = \int_a^b \rho(x,t)\,dx + \int_t^{t+h} \Phi(a,\tau)\,d\tau - \int_t^{t+h} \Phi(b,\tau)\,d\tau \;\Rightarrow \left(\dots \lim_{h\to 0^+}\right)$$

$$\Rightarrow \int_a^b \rho_t(x,t)\,dx = -\int_a^b \phi_x(x,t)\,dx \;\rightarrow\; \Big|\; \rho_t = -\phi_x$$

- Model for the (mass) flux?

$\Phi(x,t) = c\,\rho(x,t)$ ~ "The fastest the fluid moves, the more mass comes in".

Intuition: $\Phi \sim \dfrac{\text{mass}}{\text{time} \times \text{area}} \underset{\substack{\uparrow \\ 1\text{-d.}}}{\sim} \dfrac{\text{mass}}{\text{time}}$



mass in $= \rho\,\Delta x = \rho\,c\,h \Rightarrow \Phi = c\rho$

$c = \dfrac{\Delta x}{h}$

In summary, the (linear) transport equation is:

$$\boxed{\rho_t + c\,\rho_x = 0} \quad \text{Transport equation.}$$

- Remark: One can also model situations where the velocity of the fluid depends on the density of the pollutant $c = c(\rho)$, obtaining a nonlinear transport PDE.

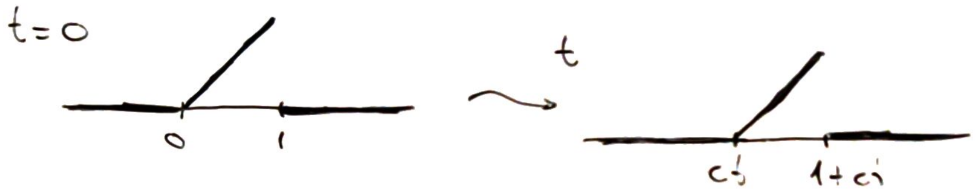
• We already know how to solve the (linear) transport equation:

$$\left.\begin{array}{l}\rho_t + c\rho_x = 0\\ \rho(x,0) = \rho_0(x)\end{array}\right\}$$ Characteristics curves: $x(t) = ct + x_0 \Rightarrow$

$$\Rightarrow \frac{d}{dt}\left(\rho(ct+x_0,t)\right) = 0 \Rightarrow \rho(ct+x_0,t) = cte = \rho(x_0,0) = \rho_0(x_0) \Rightarrow$$

$$\Rightarrow \rho(x,t) = \rho_0(x-ct) \;//.$$

For example, if $\rho_0(x) = \begin{cases} x, & 0<x<1 \\ 0, & \text{otherwise} \end{cases}$, then



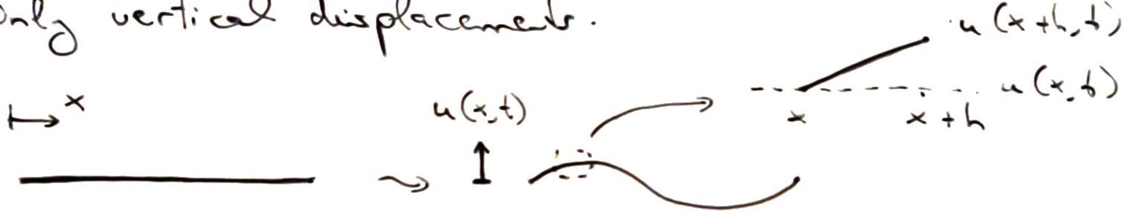
That is, the initial profile is simply transported to the right with constant velocity c.

3] Derivation of the wave equation.

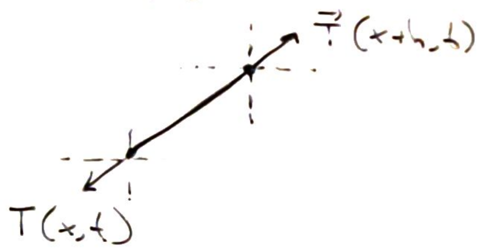
- One-dimensional case: Vibrating string.

Consider a string on a plane. Assume that:

1) It is very thing (1-d) and homogeneous ($\rho(x) = cte$).
2) Completely flexible ($\Rightarrow$ forces only in tangential direction).
3) Small displacements and velocities.
4) Only vertical displacements.



Let's apply Newton's law to a "very small" piece:



[Horizontal] $\rightarrow$ $0 = T_1(x+h) - T_1(x)$

[Vertical] $\rightarrow$ $\underbrace{\rho h}_{\rho\sqrt{h^2 + \mathcal{O}(u^2)} \approx \rho h} u_{tt}(x + h/2) = T_2(x+h) - T_2(x)$

• T_1, T_2 : tangent vector $\vec{\tau}(x) = (1, u_x(x,t))$

Thus, $\vec{T}(x) = T\,\vec{s}(x) = T \dfrac{\vec{\tau}(x)}{\sqrt{1+(u_x(x,t))^2}}$ and so

$$T_1 = \vec{T}\cdot(1,0) = T\frac{1}{\sqrt{1+u_x^2}} \approx T$$

$$T_2 = \vec{T}\cdot(0,1) = T\frac{u_x}{\sqrt{1+u_x^2}} \approx T u_x$$

$$\frac{1}{\sqrt{1+u_x^2}} = 1 - \frac{u_x^2}{2} + O(u_x^4) \approx 1 \quad \text{if } |u_x| = \varepsilon \ll 1.$$

Taylor expansion

Therefore,

[Hor.] $\rightarrow T(x+h) = T(x) \Rightarrow T \equiv \text{ct}.$

[Ver.] $\rightarrow \rho h\, u_{tt}(x+h/2) = T\,(u_x(x+h) - u_x(x)) \Rightarrow$

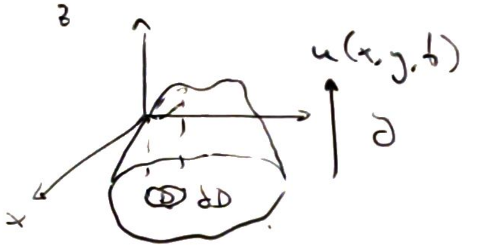
$$\Rightarrow \rho\, u_{tt}(x,t) = T\, u_{xx}(x,t) \Rightarrow$$

$$\Rightarrow \boxed{u_{tt} = c^2 u_{xx}} \quad \text{(1-d) WAVE EQUATION.}$$

$$c^2 = \frac{T}{\rho} > 0$$

(T is a modulus, ρ is the density)

• Two-dimensions: Vibrating drumhead



(same assumptions: only vertical movement, ...)

Surface that describes the drumhead: $(x, y, u(x,y))$.

Tangent vectors: $(1, 0, u_x)$
$(0, 1, u_y)$.

Thus, tangent force: $\vec{T} \approx T_1 (1, 0, u_x) + T_2 (0, 1, u_y)$.

(again $\sqrt{1+u_x^2} \approx 1$, $\sqrt{1+u_y^2} \simeq 1$).

Newton's law: (on an arbitrary region D)

$$\iint_D \rho(x,y)\, u_{tt}(x,y)\, dx\, dy = \int_{\partial D} \vec{T} \cdot (0,0,1)\, ds \qquad [\text{Vertical}].$$

The horizontal (x-y) balance gives that (not easy)

$T = |\vec{T}| = \sqrt{T_1^2 + T_2^2} = \text{cte}$, $(T_1, T_2) = T\,\vec{n}$ with

$\vec{n} \equiv$ outward unit normal to ∂D.

Then, we have that

$$\iint_D \rho(x,y)\, u_{tt}(x,y)\,dx\,dy = \int_{\partial D} (T_1 u_x + T_2 u_y)\,ds =$$

$$= \int_{\partial D} (T_1, T_2)\cdot \nabla u\, ds = \int_{\partial D} T\,\vec{n}\cdot \nabla u\, ds.$$

Finally, we can apply Green's theorem to find that (*)

$\iint_D \rho\, u_{tt}\,dx\,dy = \int_D \nabla\cdot(T\nabla u)\,dx\,dy$, and since D is arbitrary,

$$\rho u_{tt} = \nabla\cdot(T\nabla u) = T\Delta u \rightarrow \boxed{u_{tt} = c^2 \Delta u}$$ WAVE EQUATION in $\mathbb{R}^n$.

$$\left(\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial^2 x_i} \right)$$

(*) $$\iint_D \nabla\cdot\vec{g}\,dx\,dy = \int_{\partial D} \vec{g}\cdot\vec{n}\,ds$$

4] Laplace equation.

In a stationary situation, the solution doesn't depend on time.

We can imagine a river flowing in such a way that two pictures at different times look exactly the same.

This can be translated into saying that $\partial_t \equiv 0$.

Of course, this does not mean that the river is not moving, but only that that movement is independent of time.

The heat and wave equation become the Laplace equation in the stationary case:

$$\left.\begin{array}{ll} \text{HEAT EQ:} & u_t = c^2 \Delta u \\ \text{WAVE EQ:} & u_{tt} = c^2 \Delta u \end{array}\right\} \xrightarrow{\partial_t \equiv 0} \boxed{\Delta u = 0} \begin{array}{l} \text{LAPLACE} \\ \text{EQUATION.} \end{array}$$

• Solutions to $\Delta u = 0$ are called <u>harmonic functions</u>.

- Boundary and initial conditions.

In general we need to give initial conditions, that is, the value of our unknown at $t=0$ (also of its "velocity" if there are second order derivatives in time), and boundary conditions.

Ex: $u_{tt} - u_{xx} = 0 \qquad t>0,\ 0<x<L$

$$\left.\begin{aligned} u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned}\right\} \quad 0 \le x \le L$$

$$u(0,t) = 0 = u(L,t) = 0 \qquad t \ge 0 \qquad \text{(fixed ends of the string)}$$

The boundary conditions are typically classified in three types:

(D) Dirichlet condition: u is given

(N) Neumann condition: $n \cdot \nabla u$ is given (normal derivative, $\frac{\partial u}{\partial n}$)

(R) Robin condition: $n \cdot \nabla u + a(x,t)u$ is given

Ex: Heat equation

Neumann? → insulation.

$$\left.\begin{aligned} u_t &= u_{xx} \quad \text{in } D,\ t>0 \\ u(x,0) &= f(x), \quad x \in D \\ u(x,t) &= 10, \quad x \in \partial D,\ t \ge 0 \end{aligned}\right\}$$

$\partial D \equiv$ boundary of the cylinder

→ temperature of the walls fixed at 10.