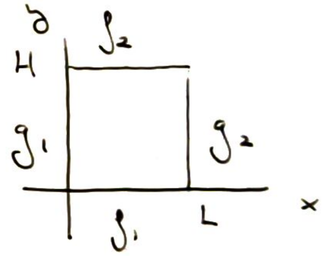
MATH 241 LECTURE 6	: Method of Separation of Variables III (~ Sec. 2.5)

[Section 2.5.1] Laplace equation inside a rectangle.

It can be shown that the equilibrium (or steady-state) temperature of a two-dimensional flat region satisfies the so-called Laplace equation: (no heat generation).

$$\Delta u(x,y) = 0 \rightsquigarrow u_{xx}(x,y) + u_{yy}(x,y) = 0.$$

As an exercise, we want to find the equilibrium temperature in a rectangular region when we are given the temperature at the boundary (four sides):



PDE: $u_{xx} + u_{yy} = 0, \quad 0 < x < L, \ 0 < y < H,$

B.C.:

$$u(0,y) = g_1(y), \qquad 0 < y < H,$$
$$u(L,y) = g_2(y),$$
$$u(x,0) = f_1(x), \qquad 0 < x < L.$$
$$u(x,H) = f_2(x).$$

Remark: For this time-independent problem, we are not given an initial condition. If the B.C. were all insulation (for example), then the solution wouldn't be unique (remember the 1-d version).

- We have four nonhomogeneous BC. But we can use that the problem is linear:

If
$$\left.\begin{array}{l} (u_1)_{xx} + (u_1)_{yy} = 0 \\ u_1(0,y) = g_1(y) \\ u_1(L,y) = 0 \\ u_1(x,0) = 0 \\ u_1(x,H) = 0 \end{array}\right\} \quad \left.\begin{array}{l} (u_2)_{xx} + (u_2)_{yy} = 0 \\ u_2(0,y) = 0 \\ u_2(L,y) = g_2(y) \\ u_2(x,0) = 0 \\ u_2(x,H) = 0 \end{array}\right\} \ \ldots \text{ analogous for } u_3, u_4,$$

then it is clear that $u = u_1 + u_2 + u_3 + u_4$ solves the original problem:
$$u(0,y) = g_1(y) + 0 + 0 + 0 = g_1(y),$$
$$u(L,y) = 0 + g_2(y) + 0 + 0 = g_2(y),$$
$$\ldots$$

- Thus, we need to solve independently four problems, and later add the solutions.

We will only show how to solve the 3rd one (the rest are very similar; the 1st is done in the book).

For clarity, I will denote $v(x,y) = u_3(x,y)$. Then, we have to solve

$$\left.\begin{array}{lll} \text{PDE}: & u_{xx} + u_{yy} = 0, & 0<x<L, \\ & & 0<y<H \\ \text{BC.} & u(0,y) = 0, & \\ & & 0<y<H, \\ & u(L,y) = 0, & \\ & u(x,0) = f_1(x), & \\ & & 0<x<L. \\ & u(x,H) = 0, & \end{array}\right\}$$

- Separation of variables: $u(x,y) = \phi(y)\, G(x)$.

Then, substitution in the PDE gives that:

$$\phi(y)\, G''(x) + \phi''(y)\, G(x) = 0 \Rightarrow$$

$$\Rightarrow -\frac{\phi''(y)}{\phi(y)} = \frac{G''(x)}{G(x)} = -\lambda .$$

- Two ODEs: ① $\phi''(y) = \lambda \phi(y)$

  ② $G''(x) = -\lambda G(x)$.

From the BC:

$$\left.\begin{array}{l} u(0,y) = 0 = \phi(y) G(0) \Rightarrow G(0) = 0 \\ u(L,y) = 0 = \phi(y) G(L) \Rightarrow G(L) = 0 \end{array}\right\}$$

$$u(x,H) = 0 = \phi(H) G(x) \Rightarrow \phi(H) = 0 \quad \}$$

Therefore, we have a boundary value problem for $G(x)$:

$$\left.\begin{array}{l} G''(x) + \lambda G(x) = 0 \\ G(0) = 0 = G(L) \end{array}\right\} \; 0 < x < L$$

We have solved this before. The eigenvalues/eigenfunctions are:

$$\lambda = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \ldots$$

$$G(x) = \sin\left(\frac{n\pi x}{L}\right)$$

- Now we go back to 1 with the eigenvalues that we found:

$$\left.\begin{array}{l} \phi''(y) = \left(\frac{n\pi}{L}\right)^2 \phi(y) \\ \phi(H) = 0 \end{array}\right\} \rightarrow \phi(y) = c_1 \cosh\left(\frac{n\pi y}{L}\right) + c_2 \sinh\left(\frac{n\pi y}{L}\right).$$

$$\hookrightarrow \phi(H) = 0 = c_1 \cosh\left(\frac{n\pi H}{L}\right) + c_2 \sinh\left(\frac{n\pi H}{L}\right) \Rightarrow c_2 = -c_1 \frac{\cosh\left(\frac{n\pi H}{L}\right)}{\sin\left(\frac{n\pi H}{L}\right)} \Rightarrow$$

$$\Rightarrow \phi(y) = c_1 \cosh\left(\frac{n\pi y}{L}\right) - c_1 \frac{\cosh\left(\frac{n\pi H}{L}\right)}{\sin\left(\frac{n\pi H}{L}\right)} \sinh\left(\frac{n\pi y}{L}\right) =$$

$$= \frac{c_1}{\sin\left(\frac{n\pi H}{L}\right)} \left(\cosh\left(\frac{n\pi y}{L}\right) \sin\left(\frac{n\pi H}{L}\right) - \cosh\left(\frac{n\pi H}{L}\right) \sinh\left(\frac{n\pi y}{L}\right)\right) =$$

$$= c_3 \sinh\left(\frac{n\pi (H-y)}{L}\right).$$

where we have used that $\sinh(a+b) = \sinh(a)\cosh(b) + \cosh(a)\sinh(b)$.

- The product solutions are:

$$u(x,y) = A \sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi(H-y)}{L}\right)$$

- Superposition principle:

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi (H-y)}{L}\right) \quad [S].$$

we need to impose the nonhomogeneous condition:

$$u(x,0) = f_1(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi H}{L}\right).$$

- Using the orthogonality of $\sin\left(\frac{n\pi x}{L}\right)$ over $x=0$ and $x=L$,

$$\int_0^L f_1(x) \sin\left(\frac{n\pi x}{L}\right) dx = A_n \sinh\left(\frac{n\pi H}{L}\right)\frac{L}{2} \Rightarrow$$

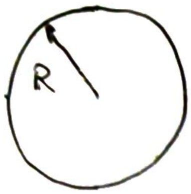
$$\Rightarrow A_n = \frac{2}{L\sinh\left(\frac{n\pi H}{L}\right)} \int_0^L f_1(x) \sin\left(\frac{n\pi x}{L}\right) dx \quad [C].$$

Thus, the solution is given by [S] with [C] //.

<u>Remark</u>: We'd need to solve 3 more problems to get the solution to the full problem in page -52-.

[Section 2.5.2] Laplace equation inside a disk.

As in the previous section, we want to study the time-independent temperature distribution in a 2-d flat region. In this case, a disk. The temperature on the boundary is given.



Polar coordinates : $u(r,\theta)$ $\left(\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right)$.

$$\Delta u(r,\theta) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

(notice that the Laplacian operator Δ, or ∇^2, is different in x-y and polar coordinates).

The problem to solve is:

$$\left.\begin{array}{lll} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, & 0 \le r < R, & \text{PDE} \\ & \theta \in [-\pi,\pi], & \\ u(R,\theta) = f(\theta), \quad -\pi \le \pi \le \pi & & \text{B.C.} \end{array}\right\}$$

We want to apply the method of separation of variables.

→ Notice that we have to impose periodic boundary conditions:

$$\left.\begin{array}{l} u(r,-\pi) = u(r,\pi) \\ u_\theta(r,-\pi) = u_\theta(r,\pi) \end{array}\right\} \text{BC.}$$

- Separation of variables: $u(r,\theta) = G(r)\Phi(\theta)$

- Substitution into PDE:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,G'(r)\Phi(\theta)\right) + \frac{1}{r^2}G(r)\Phi''(\theta) = 0 \rightarrow$$

$$\Rightarrow -\frac{r}{G(r)}\frac{d}{dr}\left(r\,G'(r)\right) = \frac{\Phi''(\theta)}{\Phi(\theta)} = -\lambda .$$

- <u>B.C.</u>:

$$u(r,-\pi) = u(r,\pi) \rightarrow \Phi(-\pi) = \Phi(\pi)$$

$$u_\theta(r,-\pi) = u_\theta(r,\pi) \rightarrow \Phi'(-\pi) = \Phi'(\pi) .$$

- ODEs: ① $\frac{r}{G(r)}\frac{d}{dr}\left(r\,G'(r)\right) = \lambda$,

② $\Phi'' + \lambda\Phi = 0$

$\Phi(-\pi) = \Phi(\pi)$

$\Phi'(-\pi) = \Phi'(\pi)$

- Eigenvalue problem (BVP): (②)

We have already solved this problem (page -48-, -49-), and (with $L=\pi$ here) obtained that

$\lambda = n^2$, $n = 1, 2, \dots$, $\qquad\qquad$ $\lambda = 0 \rightarrow \Phi(\theta) = c$.

$\Phi(\theta) = c_1\cos(n\theta) + c_2\sin(n\theta)$,

• Solution to ①:

$$\frac{d}{dr}\left(r G'(r)\right) = \lambda \frac{G(r)}{r} \to G'(r) + r G''(r) = \lambda \frac{G(r)}{r} \to$$

$$\to \left| \, r^2 G''(r) + r G'(r) - \lambda G(r) = 0 \right.$$

$\underline{\lambda = 0} \to r G'(r) = c_1 \to G(r) = c_1 \log(r) + c_2$

$\underline{\lambda = n^2 > 0} \to \boxed{r^2 G''(r) + r G'(r) - n^2 G(r) = 0}$ Euler equation

↳ Two independent solutions of the form $G(r) = r^p$:

$$r^2 p(p-1) r^{p-2} + r p r^{p-1} - n^2 r^p = 0 \to$$

$$\to p(p-1) + p - n^2 = 0 \to p^2 = n^2 \to p = \pm n.$$

Thus,
$$G(r) = c_3 r^n + \frac{c_4}{r^n}.$$

Additional BC: For physical reasons, we impose that the solution is bounded at the origin:

$$u(0,\theta) = G(0)\Phi(\theta) < \infty \to G(0) < \infty.$$

Therefore, $c_1 = 0 = c_4$ and thus $G(r) = c r^n$, $n = 0, 1, \ldots$

- Product solutions: $u(r,\theta) = c_1 r^n \cos(n\theta) + c_2 r^n \sin(n\theta)$,

- Superposition principle

$$u(r,\theta) = \sum_{n=0}^{\infty} A_n r^n \cos(n\theta) + \sum_{n=1}^{\infty} B_n r^n \sin(n\theta).$$

- Finally, we use the nonhomogeneous B.C. to determine A_n, B_n:

$$u(R,\theta) = f(\theta) = \sum_{n=0}^{\infty} A_n R^n \cos(n\theta) + \sum_{n=1}^{\infty} B_n R^n \sin(n\theta).$$

Using the orthogonality property of $\sin(n\theta), \cos(n\theta)$, we obtain that

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta,$$

$$A_n = \frac{1}{\pi R^n}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\,d\theta \quad B_n = \frac{1}{\pi R^n}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\,d\theta, \quad n = 1,2,\ldots$$