MATH 241 LECTURE 7

: Fourier Series (Sect. 3.1-3.3)

¶ Last day: Method of Separation of Variables (Laplace equation)

Exercise: Solve Laplace equation inside a 90° sector of a circular annulus $a<r<b$, $0<\theta<\frac{\pi}{2}$, subject to

$$u(r,0)=0, \quad u(r,\tfrac{\pi}{2})=0, \quad a<r<b, \qquad u(a,\theta)=0, \quad u(b,\theta)=f(\theta), \quad 0<\theta<\tfrac{\pi}{2}.$$

Sol: Eq. is $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0$

- Separation of variables $u(r,\theta)=W(r)Z(\theta)$.
- BC: $u(r,0)=0=W(r)Z(0) \Rightarrow Z(0)=0$

  $u(r,\frac{\pi}{2})=0=W(r)Z(\frac{\pi}{2}) \Rightarrow Z(\frac{\pi}{2})=0$

  $u(a,\theta)=0=W(a)Z(\theta) \Rightarrow W(a)=0$

- ODEs: $\frac{1}{r}Z(\theta)\frac{d}{dr}\left(rW'(r)\right)+\frac{1}{r^2}W(r)Z''(\theta)=0 \Rightarrow$

$$\Rightarrow \frac{r}{W(r)}\frac{d}{dr}\left(rW'(r)\right)=-\frac{Z''(\theta)}{Z(\theta)}=\lambda.$$

- BVP (eigenvalues) for $Z(\Theta)$:

$$\left.\begin{array}{l} Z''(\Theta) + \lambda Z(\Theta) = 0 \\ Z(0) = 0 \\ Z\left(\frac{\pi}{2}\right) = 0 \end{array}\right\} \rightarrow \left| \begin{array}{l} \lambda = \left(\frac{n\pi}{\pi/2}\right)^2 = 4n^2, \; n = 1, 2, \dots \\ \\ Z(\Theta) = \sin(2n\Theta) \end{array}\right.$$

- For $\lambda = 4n^2, n = 1, 2, \dots$ ODE for $W(r)$:

$$\frac{r}{W(r)} \frac{d}{dr}\left(r W'(r)\right) = 4n^2 \Rightarrow r\, W'(r) + r^2 W''(r) - 4n^2 W(r) = 0$$

Let $W(r) = r^p \Rightarrow r p r^{p-1} + r^2 p(p-1) r^{p-2} - 4n^2 r^p = 0 \Rightarrow$

$$\Rightarrow p = \pm 2n$$

Therefore, $W(r) = c_1 r^{2n} + c_2 \frac{1}{r^{2n}}$.

B.C. for $W(r)$: $W(a) = 0 \Rightarrow c_1 a^{2n} + c_2 \frac{1}{a^{2n}} = 0 \Rightarrow$

$$\Rightarrow c_2 = -c_1 a^{4n}, \text{ so}$$

$$W(r) = c_1 r^{2n} - c_1 \frac{a^{4n}}{r^{2n}} = c_1 \left(r^{2n} - \frac{a^{4n}}{r^{2n}}\right).$$

- Product solution:

$$u(r,\Theta) = c_1 \left(r^{2n} - \frac{a^{4n}}{r^{2n}}\right) \sin(2n\Theta)$$

• Superposition principle:

$$u(r,\theta) = \sum_{n\geq 1} B_n \left(r^{2n} - \frac{a^{4n}}{r^{2n}} \right) \sin(2n\theta),$$

with the nonhomogeneous BC

$$u(b,\theta) = f(\theta) = \sum_{n\geq 1} B_n \left(b^{2n} - \frac{a^{4n}}{b^{2n}} \right) \sin(2n\theta).$$

Thus, using the orthogonality property of $\sin(2n\theta)$ over $[0, \frac{\pi}{2}]$, we obtain the values of B_n:

$$B_n = \frac{b^{2n}}{b^{4n} - a^{4n}} \frac{4}{\pi} \int_0^{\pi/2} f(\theta) \sin(2n\theta)\, d\theta.$$

[Chapter 3] Fourier Series.

<u>Definition</u>: The Fourier Series of a function $f(x)$ over the interval $-L \leq x \leq L$ is

$$F(f)(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

- We want to answer the following questions:

1) Are the coefficients well-defined?

2) Is the series "well-defined" (i.e., convergent)?

3) If the series converges, does it converge to $f(x)$? That is, is $f(x)$ equal to its Fourier series?

<u>Remark</u>: In the previous problems we assumed all these points were true.

• <u>Theorem</u>: Convergence of the Fourier series.

If $f(x)$ is piecewise smooth on $-L \le x \le L$, then the Fourier series $F(f)(x)$ converges to

1) the periodic extension of $f(x)$ where the periodic extension is continuous,
2) to the average of the two sided limits, that is,

$$\frac{1}{2}\left(f(x^+) + f(x^-)\right),$$

where the periodic extension has a jump discontinuity.

<u>Remark</u>: If we limit ourselves to $-L \le x \le L$, we have that

$$\frac{1}{2}\left(f(x^+) + f(x^-)\right) = F(f)(x),$$ and, in particular, where $f(x)$ is continuous we have $f(x) = F(f)(x)$.

<u>Remark</u>: Notice that, assuming the convergence, the definition of the coefficients is simply a consequence of the orthogonality property for $\sin\left(\frac{n\pi x}{L}\right)$, $\cos\left(\frac{n\pi x}{L}\right)$ over $-L \le x \le L$.

• <u>Example</u>: Sketch $f(x)$ and the Fourier series of $f(x)$ over $-L \leq x \leq L$. Find the Fourier series coefficients.

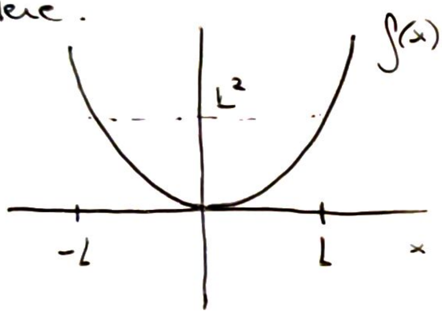
a) $f(x) = x^2$

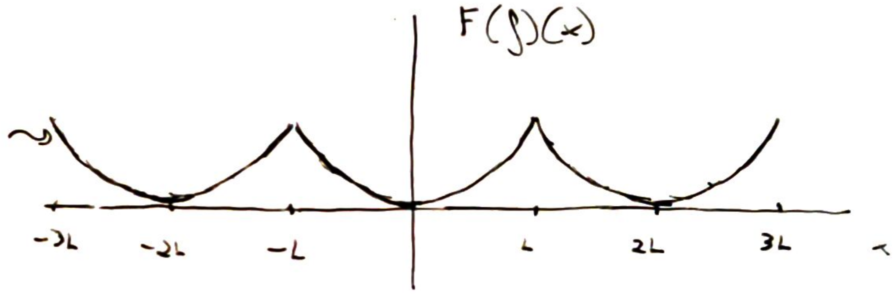
To sketch the Fourier series $F(f)(x)$ we use the previous theorem: ① Sketch $f(x)$ over $-L \leq x \leq L$.

② Sketch its periodic extension.

③ On the jump discontinuities, $F(f)(x)$ is the average.

Here.





Since the periodic extension is continuous everywhere, then $f(x) = F(f)(x)$ for all $-L \leq x \leq L$.

Coefficients: $a_0 = \dfrac{1}{2L}\displaystyle\int_{-L}^{L} x^2\, dx = \dfrac{1}{L}\int_{0}^{L} x^2\, dx = \dfrac{L^2}{3}$.

$$a_n = \frac{1}{L}\int_{-L}^{L} x^2 \cos\left(\frac{n\pi x}{L}\right) dx, \quad \left.\begin{array}{l} u = x^2 \longrightarrow du = 2x\,dx \\ dv = \cos\left(\frac{n\pi x}{L}\right) dx \to v = \frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right) \end{array}\right\} \to$$

$$a_n = \frac{-1}{L}\int_{-L}^{L} \frac{L}{n\pi} 2x \sin\left(\frac{n\pi x}{L}\right) dx + \frac{L}{n\pi} x^2 \cancel{\sin\left(\frac{n\pi x}{L}\right)}\Big]_{-L}^{L} =$$

$$= \left\{ \begin{array}{l} u = 2x \to du = 2dx \\ dv = \sin\left(\frac{n\pi x}{L}\right) dx \to v = -\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right) \end{array} \right\} =$$

$$= \frac{-1}{n\pi}\left(-\frac{L}{n\pi} 2x \cos\left(\frac{n\pi x}{L}\right)\Big]_{-L}^{L} + \int_{-L}^{L} \frac{L}{n\pi} 2 \cos\left(\frac{n\pi x}{L}\right) dx \right) =$$

$$= \frac{L}{(n\pi)^2}\Big(2L\cos(n\pi) + 2L\cos(n\pi)\Big) + \frac{2L}{n\pi}\frac{L}{n\pi} \cancel{\sin\left(\frac{n\pi x}{L}\right)}\Big]_{-L}^{L} =$$

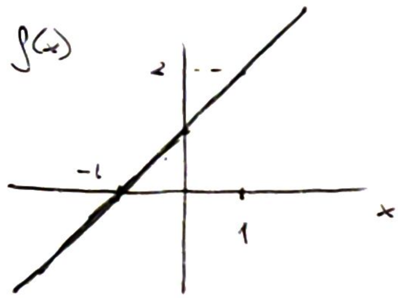
$$= \frac{4L^2}{(n\pi)^2}\cos(n\pi) = \begin{cases} \dfrac{4L^2}{(n\pi)^2} & n \text{ even}, \\[2ex] -\dfrac{4L^2}{(n\pi)^2} & n \text{ odd}. \end{cases}$$

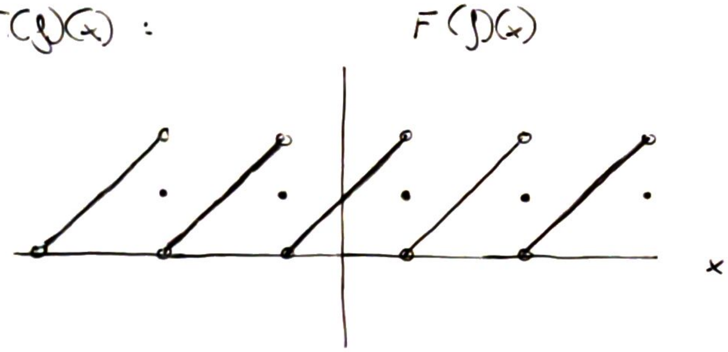
$$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\underbrace{x^2}_{\text{even}} \underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{\text{odd}}}_{\text{odd}} dx = 0 \;!!$$

<u>Remark</u>: $f(x) = x^2$ is an <u>even</u> function $\to$ We see that its Fourier series only has <u>cosines</u>.

b) $f(x) = 1 + x$, $L = 1$.

Sketch of $f(x)$ and $F(f)(x)$:





Here we have that
$$F(f)(x) = \begin{cases} \vdots \\ 1+x & \text{in } -1 < x < 1 \\ \frac{1}{2} & \text{for } x = -1, 1 \\ \vdots \end{cases}$$

Coefficients:

$$a_0 = \frac{1}{2}\int_{-1}^{1}(1+x)\,dx = \frac{1}{2}(2+0) = 1.$$

$$a_n = \int_{-1}^{1}(1+x)\cos(n\pi x)\,dx = 2\int_0^1 \cos(n\pi x)\,dx = 2\,\frac{1}{n\pi}\sin(n\pi x)\Big]_0^1 = 0$$

$$b_n = \int_{-1}^{1}(1+x)\sin(n\pi x)\,dx = \int_{-1}^{1} x\sin(n\pi x)\,dx = 2\int_0^1 x\sin(n\pi x)\,dx =$$

$$= \left\{ \begin{array}{l} u = x \to du = dx \\ dv = \sin(n\pi x)\,dx \to v = \frac{-1}{n\pi}\cos(n\pi x) \end{array} \right\} =$$

$$= \frac{-2}{n\pi}x\cos(n\pi x)\Big]_0^1 + \frac{2}{n\pi}\int_0^1 \cos(n\pi x)\,dx =$$

$$= \frac{-2}{n\pi}\cos(n\pi) + \frac{2}{n\pi}\frac{1}{n\pi}\cancel{\sin(n\pi x)}\Big]_0^1 = \frac{-2}{n\pi}\cos(n\pi) = \begin{cases} \frac{2}{n\pi} & n \text{ odd}. \\ \frac{-2}{n\pi} & n \text{ even} \end{cases}$$

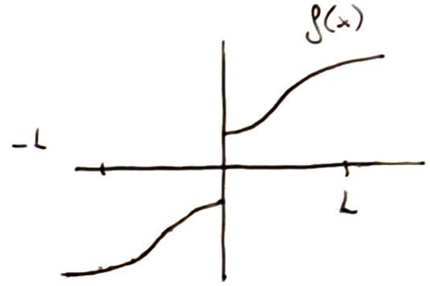
- <u>Definition</u>: The Fourier Sine Series of $f(x)$ over the interval $\underline{0 \le x \le L}$ is

$$S(f)(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \text{ with } b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

<u>Remark</u>: Notice the relationship with the (full) Fourier series:

Given $f(x)$ over $0 \le x \le L$, we can define its odd extension over $-L \le x \le L$ as follows

$$f_{odd}(x) = \begin{cases} f(x) & 0 \le x \le L \\ -f(-x) & -L \le x < 0 \end{cases}$$



Then, the (full) Fourier series of $f_{odd}(x)$ was

$$F(f_{odd})(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \text{ with}$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f_{odd}(x)\,dx = 0,$$

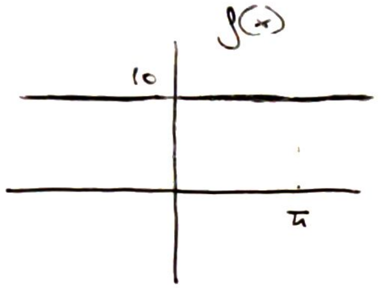
$$a_n = \frac{1}{L}\int_{-L}^{L} f_{odd}(x)\cos\left(\frac{n\pi x}{L}\right)dx = 0,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f_{odd}(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

That is, the Fourier Sine series of a function $f(x)$ over $0 \le x \le L$ is simply the Fourier series of the odd extension of $f(x)$ in $-L \le x \le L$.

<u>Example:</u> Sketch, and find the coefficients, of the Fourier sine series over $0 \le x \le \pi$ of $f(x) = 10$.

Solution:



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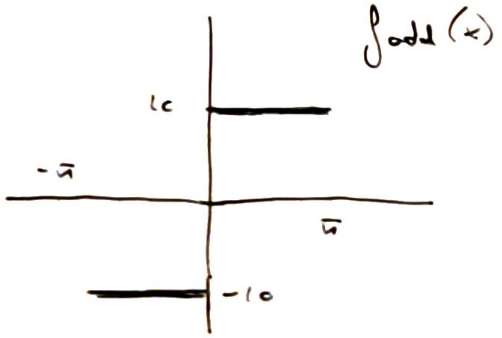

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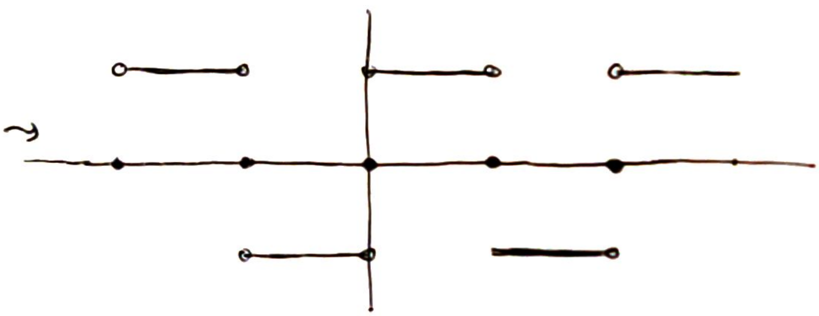


$S(f)(x) \equiv F(f_{odd})(x)$.

Coefficients: $S(f)(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$.

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx = \frac{20}{\pi}\int_0^{\pi}\sin(nx)\,dx = \frac{-20}{n\pi}\cos(nx)\Big]_0^{\pi} =$$

$$= \frac{-20}{n\pi}\left(\cos(n\pi) - 1\right) = \begin{cases} \frac{40}{n\pi} & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even.} \end{cases}$$

<u>Remark</u>: Notice that

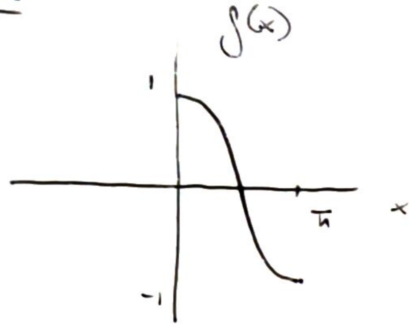
$$f(x) = 1 = \sum_{n=1}^{\infty} b_n \sin(nx) \quad \text{in } 0 < x < \pi, \text{ but it is not}$$

true at the boundary points.

- <u>Example</u>: Find the coefficients and sketch the sine Fourier series over $0 \le x \le \pi$ of $f(x) = \cos(x)$.

<u>Sol</u>:

$$f_{odd}(x) = \begin{cases} \cos(x) & 0 \le x \le \pi \\ -\cos(x) & -\pi \le x < 0 \end{cases}$$



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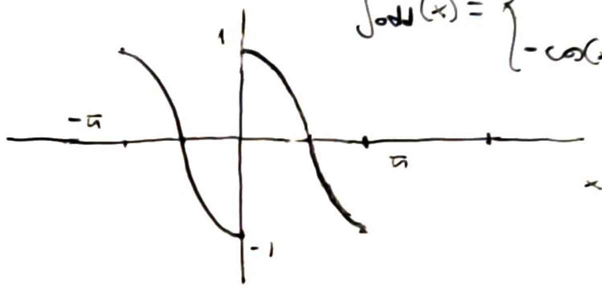
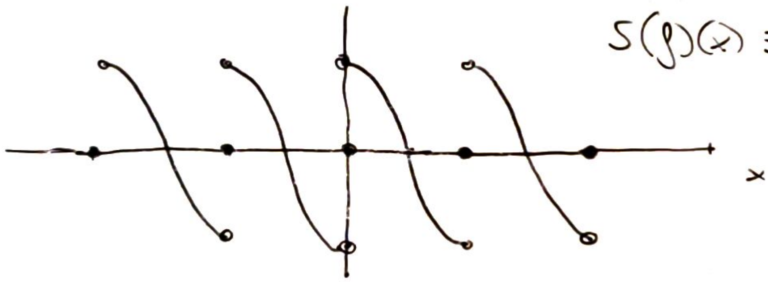


$\rightarrow$

$$S(f)(x) \equiv F(f_{odd}(x)).$$



$$S(f)(x) = \sum_{n=1}^{\infty} b_n \sin(nx), \text{ with}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} \cos(x)\sin(nx)\,dx \ne 0 \,!!$$

($\cos\left(\frac{n\pi x}{L}\right), \sin\left(\frac{m\pi x}{L}\right)$ are orthogonal over $\underline{-L \le x \le L}$)