MATH 241 LECTURE 8 : Fourier Series II.

• <u>Definition</u>: The Fourier Cosine Series of $f(x)$ over the interval $\underline{0 \le x \le L}$ is

$$C(f)(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right), \text{ with } \quad A_0 = \frac{1}{L}\int_0^L f(x)\,dx,$$

$$(n \neq 0) \quad A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx.$$

<u>Remark</u>: Analogously to the Sine Fourier series, the Cosine Fourier series over $0 \le x \le L$ of $f(x)$ is the same as the (full) Fourier series over $-L \le x \le L$ of the <u>even extension</u> of $f(x)$ to $-L \le x \le L$,

$$f_{even}(x) = \begin{cases} f(x) & 0 \le x \le L, \\ f(-x) & -L \le x < 0. \end{cases}$$

Indeed, $F(f_{even})(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right).$

with

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f_{even}(x)\,dx = \frac{1}{L}\int_0^L f(x)\,dx,$$

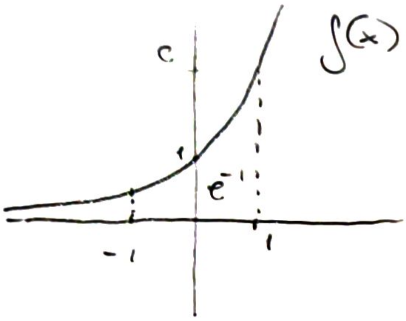
$$a_n = \frac{1}{L}\int_{-L}^{L} f_{even}(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx,$$

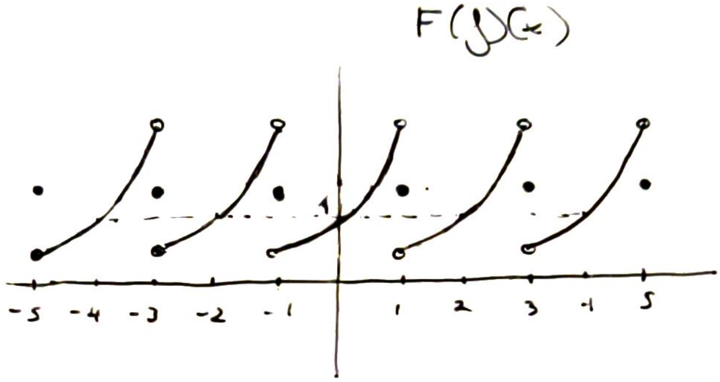
$$b_n = \frac{1}{L}\int_{-L}^{L} f_{even}(x)\sin\left(\frac{n\pi x}{L}\right)dx = 0.$$

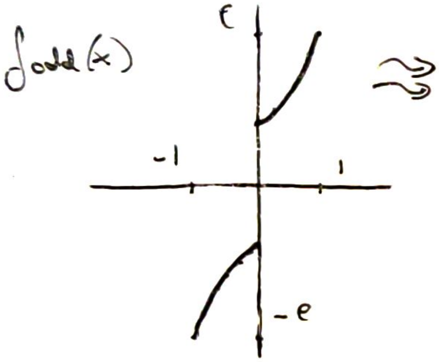
• <u>Example</u>: Let $f(x) = e^x$.

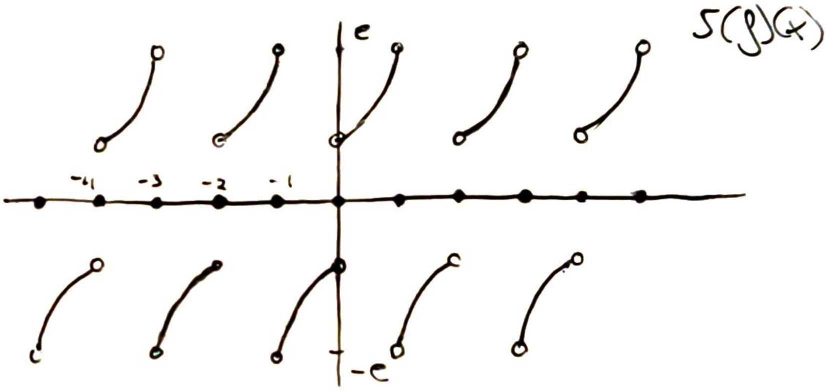
Sketch $f(x)$, the Fourier series of $f(x)$ (over $[-1,1]$), the Sine Fourier series (over $[0,1]$), and the Fourier cosine series (over $[0,1]$).

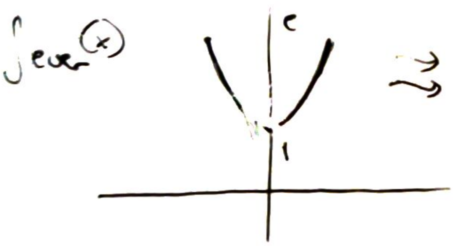
<u>Sol.</u>

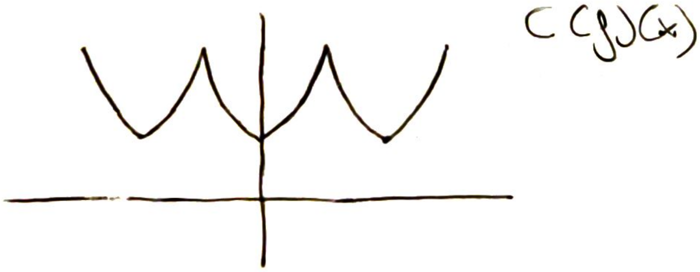








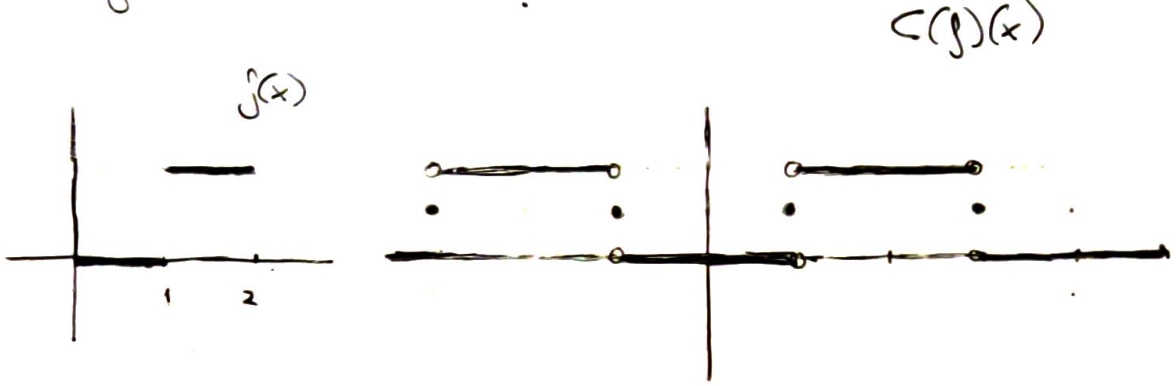




• Exercise: Consider the function $f(x) = \begin{cases} 0 & x < 1, \\ 1 & x > 2. \end{cases} \quad x \in [0,2]$

Find the Fourier Cosine series. To which value does the series converges to at $x = 1$?

Sol:



$$C(f)(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right), \qquad L = 2 \text{ in this problem.}$$

$$A_0 = \frac{1}{2}\int_0^2 f(x)\,dx = \frac{1}{2}\int_1^2 1\,dx = \frac{1}{2},$$

$$A_n = \frac{2}{2}\int_0^2 f(x)\cos\left(n\frac{\pi}{2}x\right)dx = \int_1^2 \cos\left(n\frac{\pi}{2}x\right)dx =$$

$$= \frac{2}{\pi}\sin\left(n\frac{\pi}{2}x\right)\Big]_1^2 = \frac{2}{\pi}\left(\cancel{\sin(n\pi)} - \sin\left(n\frac{\pi}{2}\right)\right) =$$

$$= \begin{cases} 0 & n \text{ even} \\ \frac{2}{\pi}(-1)^k & n = 2k-1 \text{ (odd)}. \end{cases}$$

Thus,

$$C(f)(x) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2}{\pi}(-1)^k \cos\left(\frac{2k-1}{2}\pi x\right).$$

$$C(f)(1) = \frac{1}{2}.$$

[Section 3.6] Complex Form of Fourier Series.

Recall the expression of the Fourier series over $[-L,L]$ of $f(x)$:

$$\mathcal{F}(f)(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right).$$

Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ imply that

$$\cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i},$$

so we can write

$$\mathcal{F}(f)(x) = a_0 + \sum_{n=1}^{\infty} a_n\frac{1}{2}\left(e^{\frac{n\pi x}{L}i} + e^{-\frac{n\pi x}{L}i}\right) + \frac{b_n}{2i}\left(e^{\frac{n\pi x}{L}i} - e^{-\frac{n\pi x}{L}i}\right) =$$

$$= a_0 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n - ib_n)e^{\frac{n\pi x}{L}i} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n+ib_n)e^{-\frac{n\pi x}{L}i} =$$

$m=-n$

$$= a_0 + \frac{1}{2}\sum_{m=-\infty}^{-1}(a_{-m} - ib_{-m})e^{-\frac{m\pi x}{L}i} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n+ib_n)e^{\frac{-n\pi x}{L}i}.$$

Since $$a_{-m} = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{-m\pi x}{L}\right)dx = a_m,$$

$$b_{-m} = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{-m\pi x}{L}\right)dx = -b_m,$$

$$\mathcal{F}(f)(x) = a_0 + \frac{1}{2}\sum_{m=-\infty}^{-1}(a_m + ib_m)e^{-\frac{m\pi x}{L}i} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n+ib_n)e^{\frac{-n\pi x}{L}i} =$$

$$= a_0 + \frac{1}{2}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (a_n + i b_n)\, e^{-\frac{n\pi x}{L} i}.$$

Defining $c_0 = a_0$, $c_n = \dfrac{a_n + i b_n}{2}$,

$$\boxed{F(f)(x) = \sum_{n=-\infty}^{\infty} c_n e^{-in\pi x/L}}$$ Complex form of the Fourier series of $f(x)$.

• <u>Remark</u>: From the definition above, one can find that

$$\boxed{c_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{in\pi x/L}\, dx}$$

→ We are going to obtain this formula using an orthogonality property instead:

$$\int_{-L}^{L} e^{im\pi x/L} e^{-in\pi x/L}\, dx = \begin{cases} 0 & n\neq m \\ 2L & n = m. \end{cases}$$

Now, assume $f(x) = F(f)(x)$ and multiply both sides by $e^{im\pi x/L}$, and integrate over $[-L, L]$:

$$\int_{-L}^{L} f(x) e^{im\pi x/L}\, dx = \sum_{n=-\infty}^{\infty} c_n \int_{-L}^{L} e^{im\pi x/L} e^{-in\pi x/L}\, dx \Rightarrow$$

$$\Rightarrow \int_{-L}^{L} f(x) e^{im\pi x/L}\, dx = 2L c_m \quad //.$$

• [Additional problem] [Chapter 2, MSV, Laplace equation].

Solve the following problem using separation of variables:

$$\left.\begin{array}{l} u_{xx} + u_{yy} = 0, \quad 0 < x < \infty, \ 0 < y < H, \\ u_y(x,0) = 0 \\ u_y(x,H) = 0 \\ u_x(0,y) = f(y) \end{array}\right\}$$

Questions: 1) Show that there exists a solution if and only if $\int_0^H f(y)dy = 0$.

2) Is the solution uniquely defined?

Sol:

Let $u(x,y) = F(x)G(y) \rightarrow \frac{-F''(x)}{F(x)} = \frac{G''(y)}{G(y)} = -\lambda$.

BC:

$u_y(x,0) = 0 \rightarrow G'(0) = 0,$

$u_y(x,H) = 0 \rightarrow G'(H) = 0,$

ODE: ① $\left.\begin{array}{l} G''(y) + \lambda G(y) = 0 \\ G'(0) = 0 = G'(H) \end{array}\right\} \rightarrow \begin{array}{l} \lambda = \left(\frac{n\pi}{H}\right)^2, \quad n = 0, 1, \dots \\ G(y) = \cos\left(\frac{n\pi y}{H}\right). \ n = 0, 1, \dots \end{array}$

③ $F''(x) - \left(\frac{n\pi}{H}\right)^2 F(x) = 0$

$n = 0 \rightarrow F(x) = c_1 x + c_2$

$n \neq 0 \rightarrow F(x) = c_3 e^{\frac{n\pi x}{H}} + c_4 e^{-\frac{n\pi x}{H}}$

Remark: Physically valid solutions should be bounded as $x \to +\infty$. Therefore, we take $c_1 = c_3 = 0$.

So, $F(x) = e^{-\frac{n\pi x}{H}}$, $n = 0, 1, \ldots$

• Product solution: $u(x,y) = A \cos\left(\frac{n\pi y}{H}\right) e^{-\frac{n\pi x}{H}}$

• Superposition principle:

$$u(x,y) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi y}{H}\right) e^{-\frac{n\pi x}{H}}$$

• A_n? $\rightarrow u_x(0,y) = g(y)$

$$u_x(x,y) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi y}{H}\right) \frac{-n\pi}{H} e^{-\frac{n\pi x}{H}}$$

$$u_x(0,y) = \sum_{n=0}^{\infty} A_n \frac{-n\pi}{H} \cos\left(\frac{n\pi y}{H}\right) = g(y).$$

Using the orthogonality of cosines we find that

$$\int_0^H f(y)\cos\left(\frac{n\pi y}{H}\right)dy = A_n\frac{-n\pi}{H}\frac{H}{2}, \quad n=1,2,\ldots \Rightarrow$$

$$\Rightarrow A_n = \frac{-2}{\pi}\int_0^H f(y)\cos\left(\frac{n\pi y}{H}\right)dy, \quad n=1,2,\ldots$$

Notice what happens when $n=0$:

$$\int_0^H f(y)\,dy = \sum_{n=0}^{\infty} A_n\frac{-n\pi}{H}\int_0^H\cos\left(\frac{n\pi y}{H}\right)dy = \sum_{n=1}^{\infty} A_n\frac{-n\pi}{H}\underbrace{\cancel{\int_0^H\cos\left(\frac{n\pi y}{H}\right)dy}}_{=0}$$

so we obtain the condition we were asked to show.

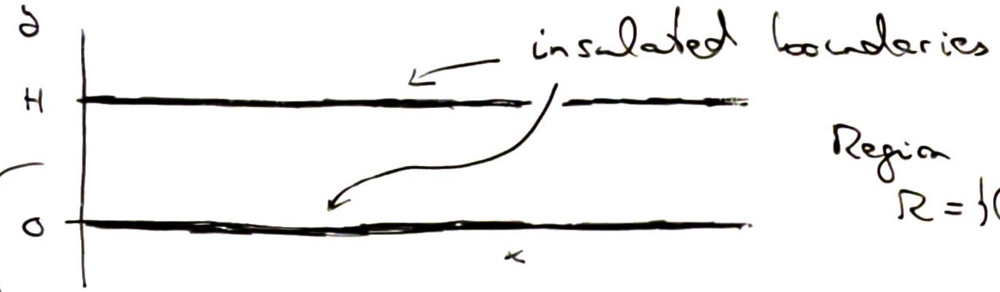
Therefore, the solution is

$$\boxed{\begin{aligned} u(x,y) &= A_0 + \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi y}{H}\right)e^{-\frac{n\pi x}{H}}, \text{ with} \\ A_n &= \frac{-2}{\pi}\int_0^H f(y)\cos\left(\frac{n\pi y}{H}\right)dy \quad \text{for } n=1,2,\ldots \end{aligned}}$$

Thus, we see that the solution <u>is not</u> uniquely defined. The constant A_0 is arbitrary.

→ What is happening?

Let's think about the physics of the problem:



Region $R = \{(x,y) : \begin{matrix} 0 \le x < \infty, \\ 0 \le y \le H \end{matrix}\}$

$u_x(0,y) = f(y)$ That is, we know the flux along that side.

→ Remember that the heat equation over a 2-d region is $u_t = k\,\Delta u = k(u_{xx} + u_{yy})$, and therefore we can think of the Laplace equation $\Delta u = 0$ as the PDE that describes the <u>equilibrium temperature</u> over that region.

↳ For an equilibrium to exists, we need that the "net flux" is zero: otherwise, the region would receive (or give) energy, and the temperature would increase (or decrease). Mathematically, the condition

$\int_0^H f(y)\,dy = 0$ precisely states that the net (or average) flux is zero along the boundary $x = 0, 0 < y < H$.

Since the B.C., with the condition $\int_0^H f(y)\,dy = 0$, state that there is no net energy going in or out, any constant temperature is an equilibrium solution (notice that there isn't any Dirichlet B.C.).

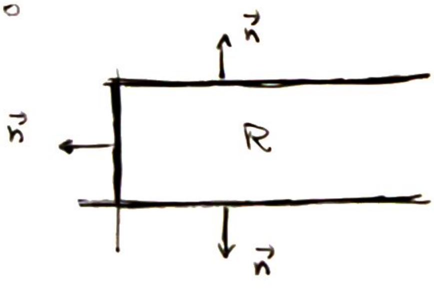
→ If we were given an initial condition, then we'd be able to determine <u>the</u> solution, by conservation of energy. Indeed, let's see it mathematically. Assume that $u(x,y,0) = g(x,y)$, with $\iint_R g(x,y)\,dx\,dy = 0$. Consider the solution to $u_t = u_{xx} + u_{yy}$ with the same B.C.

Then,

$$\frac{d}{dt}\iint_R u(x,y,t)\,dx\,dy = \iint_R \Delta u(x,y,t)\,dx\,dy = \iint_R \nabla\cdot\nabla u(x,y,t)\,dx\,dy =$$

$$= \int_{\partial R} \vec{n}\cdot\nabla u(x,y,t)\,dx\,dy = \int_0^H (-u_x(0,y))\,dy + \int_0^\infty u_y(x,H)\,dx +$$

$$+ \int_0^\infty (-u_y(x,0))\,dx = 0 \qquad \left(\text{because } \int_0^H u_x(0,y)\,dy = \int_0^H f(y)\,dy = 0\right).$$



Therefore, $\displaystyle\iint_R u(x,y,t)\,dx\,dy = \iint_R u(x,y,0)\,dx\,dy = \iint_R g(x,y)\,dx\,dy.$

This holds for all $t \geq 0$, so in particular at equilibrium:

$$\iint_R u(x,y)\,dx\,dy = \iint_R g(x,y)\,dx\,dy = 0,$$

The solution we found for the Laplace equation

We can compute the first integral:

$$\iint_R u(x,y)\,dx\,dy = \iint_R A_0\,dx\,dy + \sum_{n=1}^{\infty} A_n \underbrace{\cancel{\int_0^H \cos\left(\frac{n\pi y}{H}\right) dy}}_{=0} \int_0^{\infty} e^{\frac{-n\pi x}{H}}\,dx =$$

$$= \iint_R A_0\,dx\,dy = 0 \rightarrow A_0 = 0.$$