$\boxed{\begin{array}{l}\text{MATH 241}\\ \text{LECTURE 9}\end{array}}$: More problems.

- Solve the following problem using separation of variables:

$$\left.\begin{aligned} &u_{xx} + u_{yy} = 0, \quad 0 < x < \infty, \ 0 < y < H, \\ &u_y(x,0) = 0 \\ &u_y(x,H) = 0 \\ &u_x(0,y) = f(y) \end{aligned}\right\}$$

Questions: 1) Show that there exists a solution if and only if $\int_0^H f(y)\,dy = 0$.

2) Is the solution uniquely defined?

Sol:

Let $u(x,y) = F(x)G(y) \rightarrow \dfrac{-F''(x)}{F(x)} = \dfrac{G''(y)}{G(y)} = -\lambda$.

BC:

$u_y(x,0) = 0 \rightarrow G'(0) = 0,$

$u_y(x,H) = 0 \rightarrow G'(H) = 0,$

ODE: ① $\left.\begin{aligned} &G''(y) + \lambda G(y) = 0 \\ &G'(0) = 0 = G'(H) \end{aligned}\right\} \rightarrow \begin{aligned} &\lambda = \left(\frac{n\pi}{H}\right)^2, \quad n = 0, 1, \ldots \\ &G(y) = \cos\left(\frac{n\pi y}{H}\right), \quad n = 0, 1, \ldots \end{aligned}$

③ $F''(x) - \left(\frac{n\pi}{H}\right)^2 F(x) = 0$

$n = 0 \Rightarrow F(x) = c_1 x + c_2$

$n \neq 0 \Rightarrow F(x) = c_3 e^{\frac{n\pi x}{H}} + c_4 e^{-\frac{n\pi x}{H}}$

Remark: Physically valid solutions should be bounded as $x \to +\infty$. Therefore, we take $c_1 = c_3 = 0$.

So, $F(x) = e^{-\frac{n\pi x}{H}}$, $n = 0, 1, \ldots$

- Product solution: $u(x,y) = A \cos\left(\frac{n\pi y}{H}\right) e^{-\frac{n\pi x}{H}}$

- Superposition principle:

$$u(x,y) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi y}{H}\right) e^{-\frac{n\pi x}{H}}.$$

- A_n? $\rightarrow u_x(0,y) = g(y)$

$$u_x(x,y) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi y}{H}\right)\left(\frac{-n\pi}{H}\right) e^{-\frac{n\pi x}{H}},$$

$$u_x(0,y) = \sum_{n=0}^{\infty} A_n \frac{-n\pi}{H} \cos\left(\frac{n\pi y}{H}\right) = g(y).$$

Using the orthogonality of cosines, we find that

$$\int_0^H f(y)\cos\left(\frac{n\pi y}{H}\right)dy = A_n \frac{-n\pi}{H}\frac{H}{2} \quad , n=1,2,\ldots \Rightarrow$$

$$\Rightarrow A_n = \frac{-2}{\pi}\int_0^H f(y)\cos\left(\frac{n\pi y}{H}\right)dy \quad , n=1,2,\ldots$$

Notice what happens when $n=0$:

$$\int_0^H f(y)\,dy = \sum_{n=0}^{\infty} A_n \frac{-n\pi}{H}\int_0^H \cos\left(\frac{n\pi y}{H}\right)dy = \sum_{n=1}^{\infty} A_n \frac{-n\pi}{H}\underbrace{\int_0^H \cos\left(\frac{n\pi y}{H}\right)dy}_{=0}$$

so we obtain the condition we were asked to show.

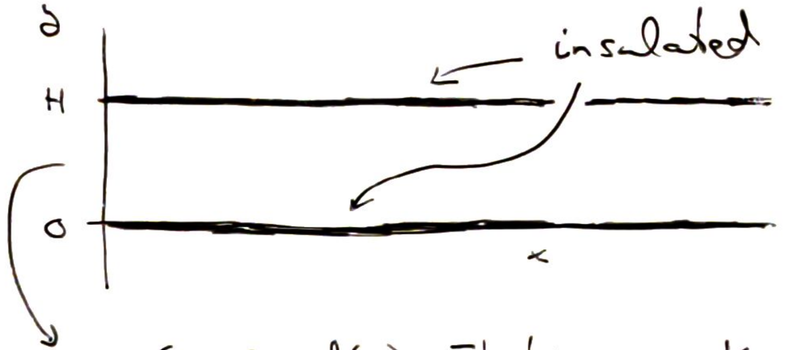
Therefore, the solution is

$$\boxed{\begin{aligned} u(x,y) &= A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi y}{H}\right)e^{-\frac{n\pi x}{H}}, \text{ with} \\ A_n &= \frac{-2}{\pi}\int_0^H f(y)\cos\left(\frac{n\pi y}{H}\right)dy \quad \text{for } n=1,2,\ldots \end{aligned}}$$

Thus, we see that the solution is not uniquely defined.
The constant A_0 is arbitrary.

→ What is happening?

Let's think about the physics of the problem:



Region
$R = \{(x,y) : \begin{matrix} 0 \le x < \infty, \\ 0 \le y \le H \end{matrix}\}$

$u_x(0,y) = f(y)$ That is, we know the flux along that side.

→ Remember that the heat equation over a 2-d region is $u_t = k\,\Delta u = k(u_{xx} + u_{yy})$, and therefore we can think of the Laplace equation $\Delta u = 0$ as the PDE that describes the <u>equilibrium temperature</u> over that region.

↳ For an equilibrium to exists, we need that the "net flux" is zero: otherwise, the region would receive (or give) energy, and the temperature would increase (or decrease). Mathematically, the condition

$\int_0^H f(y)\,dy = 0$ precisely states that the net (or average) flux is zero along the boundary $x = 0, 0 < y < H$.

Since the B.C., with the condition $\int_0^H f(y)\,dy = 0$, state that there is no net energy going in or out, any constant temperature is an equilibrium solution (notice that there isn't any Dirichlet B.C.).

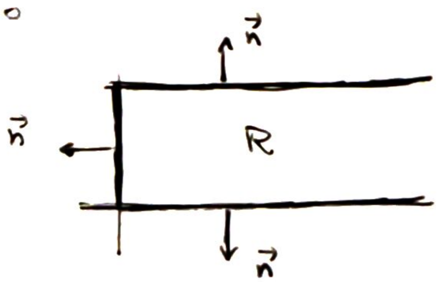
→ If we were given an initial condition, then we'd be able to determine <u>the</u> solution, by conservation of energy. Indeed, let's see it mathematically. Assume that $u(x,y,0) = g(x,y)$, with $\iint_R g(x,y)\,dx\,dy = 0$. Consider the solution to $u_t = u_{xx} + u_{yy}$ with the same B.C.

Then,

$$\frac{d}{dt}\iint_R u(x,y,t)\,dx\,dy = \iint_R \Delta u(x,y,t)\,dx\,dy = \iint_R \nabla\cdot\nabla u(x,y,t)\,dx\,dy =$$

$$= \int_{\partial R} \vec{n}\cdot\nabla u(x,y,t)\,dx\,dy = \int_0^H (-u_x(0,y))\,dy + \int_0^\infty u_y(x,H)\,dx +$$

$$+ \int_0^\infty (-u_y(x,0))\,dx = 0 \qquad \left(\text{because } \int_0^H u_x(0,y)\,dy = \int_0^H f(y)\,dy = 0\right).$$



Therefore, $\displaystyle\iint_R u(x,y,t)\,dx\,dy = \iint_R u(x,y,0)\,dx\,dy = \iint_R g(x,y)\,dx\,dy.$

This holds for all $t \geq 0$, so in particular at equilibrium:

$$\iint_R u(x,y)\,dx\,dy = \iint_R g(x,y)\,dx\,dy = 0,$$

The solution we found
for the Laplace equation

We can compute the first integral:

$$\iint_R u(x,y)\,dx\,dy = \iint_R A_0\,dx\,dy + \sum_{n=1}^{\infty} A_n \underbrace{\int_0^H \cancel{\cos\left(\frac{n\pi y}{H}\right)}\,dy}_{=0} \int_0^{\infty} e^{\frac{-n\pi x}{H}}\,dx =$$

$$= \iint_R A_0\,dx\,dy = 0 \rightarrow A_0 = 0.$$

[Problem 3, pdf] Consider $\begin{cases} x^2\phi''(x) + x\phi'(x) + \lambda\phi(x) = 0, & 1<x<e, \\ \phi(1) = \phi(e) = 0. \end{cases}$

a) Find all the eigenvalues and eigenfunctions.

b) Construct a general solution to

$$x^2 u_{xx}(x,t) + x u_x(x,t) = u_t(x,t), \quad 1<x<e,\ t>0,$$

given that $u(1,t) = 0 = u(e,t)$ for $t \geq 0$.

Sol:

• $\lambda = \alpha^2 > 0$

Try with x^p: $x^2 p(p-1)x^{p-2} + x p x^{p-1} + \lambda x^p = 0 \Rightarrow$

$$\Rightarrow p^2 + \alpha^2 = 0 \Rightarrow p = \pm|\alpha| i$$

Thus, $\phi(x) = c_1 x^{|\alpha| i} + c_2 x^{-|\alpha| i} \quad (c_1, c_2 \in \mathbb{C})$.

Notice that $x^{|\alpha| i} = e^{\ln x^{|\alpha| i}} = e^{|\alpha| i \ln x} = \cos(|\alpha|\ln x) + i\sin(|\alpha|\ln x)$.

Doing the same with $x^{-|\alpha| i}$, we find that

$$\phi(x) = (a_1 + ib_1)(\cos(|\alpha|\ln x) + i\sin(|\alpha|\ln x)) +$$
$$+ (a_2 + ib_2)(\cos(|\alpha|\ln x) - i\sin(|\alpha|\ln x)), \text{ where}$$

we are writing $\begin{matrix} c_1 = a_1 + ib_1 \\ c_2 = a_2 + ib_2 \end{matrix}$, $a_1, a_2, b_1, b_2 \in \mathbb{R}$.

Since we are only interested in real solutions, we get

$$\phi(x) = (a_1 + a_2)\cos(|\alpha| \ln x) + (b_2 - b_1)\sin(|\alpha| \ln x).$$

Renaming the constant,

$$\phi(x) = c_3 \cos(|\alpha| \ln x) + c_4 \sin(|\alpha| \ln x).$$

[Remark: We don't need to show all these steps everytime.]

B.C.: $\phi(1) = 0 = c_3$

$\phi(e) = c_4 \sin(|\alpha| \ln e) = 0 \to |\alpha| = n\pi,\ n = 1, 2, \dots$

Thus, $\lambda = (n\pi)^2,\ n = 1, 2, \dots$ are eigenvalues,

with $\phi(x) = c \cdot \sin(n\pi \ln x)$ eigenfunctions.

• $\lambda = 0$: $\frac{d}{dx}(x\phi'(x)) = 0 \to x\phi'(x) = c_1 \to$

$\to \phi(x) = c_1 \ln x + c_2$

B.C.: $\left.\begin{aligned}\phi(1) &= c_2 = 0\\ \phi(e) &= c_1 = 0\end{aligned}\right\} \to \lambda = 0$ is not an eigenvalue.

• $\underline{\lambda = -\alpha^2 < 0}$: Try with $x^p \to p = \pm|\alpha|$.

$$\phi(x) = c_1 x^{|\alpha|} + c_2 x^{-|\alpha|}.$$

$\underline{BC}$: $\left.\begin{aligned} \phi(1) &= c_1 + c_2 = 0 \\ \phi(e) &= c_1 e^{|\alpha|} + c_2 e^{-|\alpha|} = 0 \end{aligned}\right\} \to c_1 = c_2 = 0.$

Therefore, there are not negative eigenvalues.

b) $x^2 u_{xx}(x,t) + x u_x(x,t) = u_t(x,t)$, $1 < x < e$, $t > 0$,

$u(1,t) = 0 = u(e,t)$, $t \geq 0$.

$\underline{Sol}$ $u(x,t) = h(t)\phi(x) \to$

$\to x^2 h(t)\phi''(x) + x h(t)\phi'(x) = h'(t)\phi(x) \to$

$$\to x^2 \frac{\phi''(x)}{\phi(x)} + x \frac{\phi'(x)}{\phi(x)} = \frac{h'(t)}{h(t)} = -\lambda \to$$

$\to$ ① $h'(t) = -h(t)$

② $x^2 \phi''(x) + x\phi'(x) + \lambda\phi(x) = 0$

$\underline{BC}$. $u(1,t) = 0 \to \phi(1) = 0$

$u(e,t) = 0 \to \phi(e) = 0$.

Thus, from part a),

$$\lambda = (n\pi)^2, \; n = 1, 2, \ldots \quad \text{with} \quad \Phi(x) = c \sin(n\pi \ln x).$$

- ① $h(t) = e^{-t}$.

- Product solution: $u(x,t) = c \sin(n\pi \ln x)\, e^{-t}$.

- Superposition principle:

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(n\pi \ln x)\, e^{-t}.$$

The B_n are undetermined since we don't have initial condition.

• <u>Problem 1</u>: Let $u(x,t)$ be the temperature in a rod, that satisfies

$$\begin{cases} u_t = 3u_{xx} + x^3, & 0<x<4,\ t>0, \\ u_x(0,t) = 1 \\ u_x(4,t) = 5 \\ u(x,0) = x. \end{cases}$$

The total energy is defined by $E(t) = \int_0^4 u(x,t)\,dx$.

Find $E(t)$.

<u>Sol</u>: Integrate the equation from $x=0$ to $x=4$:

$$\int_0^4 u_t(x,t)\,dx = 3\int_0^4 u_{xx}(x,t)\,dx + \int_0^4 x^3\,dx \Rightarrow$$

$$\Rightarrow \frac{d}{dt}\int_0^4 u(x,t)\,dx = 3\,u_x(x,t)\Big]_0^4 + \frac{x^4}{4}\Big]_0^4 \Rightarrow$$

$$\Rightarrow \frac{dE}{dt}(t) = 3(5-1) + 4^3 = 12 + 56 = 68 \Rightarrow$$

$$\Rightarrow E(t) = 68t + c.$$

Using the initial condition, we find c:

$$E(0) = c = \int_0^4 u(x,0)\,dx = \int_0^4 x\,dx = \frac{x^2}{2}\Big]_0^4 = 8.$$

Therefore,

$$E(t) = 68t + 8.$$

• <u>Problem 4</u>: Find $u(x,t)$: $u_t - u = 7x$ with $u(x,0) = 0$.

<u>Sol</u>: Notice that e^{-t} is an integrating factor:

$$\frac{\partial}{\partial t}\left(e^{-t}u(x,t)\right) = 7xe^{-t} \Rightarrow e^{-t}u(x,t) = -7xe^{-t} + c(x)$$

we are integrating in t!

(<u>Check</u>: $\frac{\partial}{\partial t}\left[e^{-t}u = -7xe^{-t} + c(x)\right] \rightsquigarrow \frac{\partial}{\partial t}(e^{-t}u) = 7xe^{-t}$)

Thus,

$$u(x,t) = -7x + c(x)e^{t}.$$

Then, $u(x,0) = -7x + c(x) = 0 \Leftrightarrow c(x) = 7x$.

In conclusion, $u(x,t) = 7x(e^{t} - 1)$.