Math 241 - Section 002 - Midterm 1 extra problems

Remark: I have only included here problems that have not been included in the homework assignments. But please notice that many of the homework problems were exam-like problems too.

I have indicated with an asterisk * those problems that I considered might look different (but not necessarily more difficult) to you (and thus are specially worth trying).

Problem 1*

Let $u(x,t)$ be the temperature in a one-dimensional rod ($0 \leq x \leq 4$), and satisfy the following initial and boundary value problem:

$$\begin{aligned} \frac{\partial u}{\partial t} &= 3\frac{\partial^2 u}{\partial x^2} + x^3, \qquad 0 < x < 4, t > 0, \\ \frac{\partial u}{\partial x}(0,t) &= 1, \\ \frac{\partial u}{\partial x}(4,t) &= 5, \\ u(x,0) &= x. \end{aligned}$$

The total thermal energy is defined by

$$E(t) = \int_0^4 u(x,t)dx.$$

Compute $E(t)$.

Note: Similar to Problem 14.

Problem 2

Part a. Compute the Fourier sine series of

$$f(x) = \begin{cases} x, & 0 \leq x < 1/2, \\ 1 - x, & 1/2 \leq x \leq 1, \end{cases}$$

on the interval $0 \leq x \leq 1$.

Part b. For which values of $x \in [0, 1]$ does the Fourier sine series converge to $f(x)$?

Problem 3

Consider the boundary value problem

$$x^2 \frac{d^2\phi}{dx^2} + x\frac{d\phi}{dx} + \lambda\phi = 0, \qquad 1 < x < e,$$
$$\phi(1) = \phi(e) = 0.$$

Part a. Find all the eigenvalues and eigenfunctions.

Part b. Construct a general solution to the PDE

$$x^2 u_{xx}(x,t) + x u_x(x,t) = u_t(x,t),$$

for $1 < x < e$ and $t > 0$, given that $u(1,t) = 0 = u(e,t)$ for all $t \geq 0$.

Problem 4*

Find a function $u(x,t)$ that satisfies $u_t - u = 7x$ with $u(x,0) = 0$.

Problem 5*

Let $\Omega \subset \mathbb{R}^2$ be a bounded region with boundary $\partial\Omega$. Suppose $u(x,y,t)$ solves

$$u_t = \Delta u + 2u + e^t \sin(x + 2y) \quad \text{in } \Omega,$$

with $u(x,y,t) = 0$ on $\partial\Omega$ and $u(x,y,0) = 0$. Suppose also that $v(x,y,t)$ solves

$$v_t = \Delta v + 2v \quad \text{in } \Omega,$$

with $v(x,y,t) = x^2 y$ on $\partial\Omega$ and $v(x,y,0) = \cos(2x)$. Find a function $w(x,y,t)$ that satisfies

$$w_t = \Delta w + 2w + 3e^t \sin(x + 2y) \quad \text{in } \Omega,$$

with $w(x,y,t) = 5x^2 y$ on $\partial\Omega$ and $w(x,y,0) = 5\cos(2x)$. You should give a formula for w in terms of u and v.

Problem 6

Suppose

$$f(x) = \begin{cases} 1, & 0 \leq x < \pi, \\ -1, & \pi \leq x \leq 2\pi. \end{cases}$$

Part a. Compute the Fourier cosine series of f on the interval $[0, 2\pi]$.

Part b. Draw the graph of the Fourier cosine series computed above for x between -2π and 4π. Mark the value of the Fourier series with an "X" at all points of discontinuity.

Problem 7

Solve the Laplace equation

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

inside a $60°$ wedge of radius 2 subject to the boundary conditions $u(r, 0) = 0$, $u(r, \pi/3) = 0$, $u(r, \theta) = f(\theta)$.

Problem 8

Find the equilibrium solution to the heat equation $u_t = 2u_{xx} + \sin x$ with $0 < x < \pi$ subject to the boundary conditions $u(0, t) = 0$, $u(\pi, t) = 5$.

Problem 9*

Solve the Laplace equation inside a rectangle $0 \le x \le L$, $0 \le y \le H$ with boundary condition $u(0, y) = 0$, $u(L, y) = 0$, $u(x, 0) - u_y(x, 0) = 0$, $u(x, H) = f(x)$.

Problem 10*

Find the equilibrium solution to the 2D heat equation $u_t = \Delta u$ on a unit disc D with insulated boundaries $\frac{\partial u}{\partial n} = 0$, and initial condition $u(r, \theta, 0) = f(r, \theta)$.

Notation: $\frac{\partial u}{\partial n} = \vec{n} \cdot \nabla u$ denotes the directional derivative of u in the (outward) normal direction.

Note: Similar to Problem 13.

Problem 11

Solve the 1D heat equation

$$u_t = 2u_{xx},$$

for $0 < x < 1$ and $t > 0$ subject to $u_x(0, t) = 0$, $u(1, t) = 0$ for all $t > 0$ and $u(x, 0) = \cos(\pi x/2) + 4\cos(5\pi x/2)$.

Problem 12

Consider the 1D heat equation

$$u_t = u_{xx} + x,$$

with $0 < x < 1$, $t > 0$, subject to

$$u_x(0,t) = 1, \qquad u_x(1,t) = \beta,$$

and $u(x,0) = f(x)$.

Part a. For what value of β is there an equilibrium solution?

Part b. Determine the equilibrium solution.

Problem 13

Solve the Laplace equation

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

in the disk $D = \{(x,y) : x^2 + y^2 \leq 4\}$ subject to

$$\frac{\partial u}{\partial r}(2,\theta) = 32\cos(4\theta) - 8\sin(2\theta).$$

Problem 14

Consider the heat equation:

$$u_t = u_{xx} + tx, \qquad 0 < x < 1, t > 0,$$

with $u_x(0,t) = 0$, $u_x(1,t) = 0$, and $u(x,0) = \frac{1}{2}x^2 - \frac{1}{3}x^3$. Define the heat energy by

$$E(t) = \int_0^1 u(x,t)dx.$$

Find $E(t)$.

Problem 15

The heat equation:

$$u_t = u_{xx} - u, \qquad 0 < x < \pi, t > 0,$$

describes the temperature distribution of a 1D rod suffering some heat loss. Find a solution to the PDE with boundary conditions

$$u_x(0,t) = u_x(\pi,t) = 0, \qquad t > 0,$$

and initial condition $u(x,0) = 5 + \cos x + \cos(3x)$.

Problem 16*

The function $u(r,\theta)$ describes the steady state temperature distribution in a thin annulus R with outer radius 2 and inner radius 1. Suppose that the heat flux across the inner boundary of R is given by $u_r(1,\theta) = 0$, and $u_r(2,\theta) = 1 + c\sin^2(3\theta)$ for the outer circle.

Part a. What must the value of the constant c? That is, what must the value of c be so that the boundary value problem below has a solution?

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$
$$u_r(1,\theta) = 0,$$
$$u_r(2,\theta) = 1 + c\sin^2(3\theta)$$

Part b. Find the solution $u(r,\theta)$ corresponding to the value of c you found in part a).

Hint: You might want to use that $\sin^2(x) = \frac{1-\cos(2x)}{2}$.

Note: Problem 18 is a similar idea (but easier).

Problem 17

If for $0 \le x \le 5$ we have that

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{5}\right) = 5x + x^2,$$

what is the value of b_3?

Problem 18

Consider an annulus with inner radius $R_1 = 2$ and outer radius $R_2 = 5$. Suppose that the heat flux at every point of the outer circle points directly out of the annulus and has magnitude 4. Also, suppose that at every point of the inner circle the flux points directly into the annulus and has the same magnitude all around the circle. What must the magnitude of this latter flux be so that the temperature of the annulus will be at equilibrium? In other words, so that there is a solution of the Laplace equation with these flux values.

Problem 19

Consider the following BVP posed for $0 < x < L$ and $t > 0$:

$$u_t = u_{xx} + 2u_x + u,$$
$$u(0,t) = 0, \qquad u(L,t) + u_x(L,t) = 0.$$

Apply the method of separation of variables to determine what ordinary differential equations are implied for the functions of x and t and what boundary conditions (if any) are necessary for each of those ODEs. You do not need to solve these ODEs.

Problem 20

The temperature of a rod is described by the following PDE and conditions:

$$u_t = u_{xx} + e^{-x}, \qquad 0 < x < 1, t > 0,$$

with $u(0,t) + 2u_x(0,t) = 0$, $u_x(1,t) = 3$, for all $t > 0$, and $u(x,0) = \sin x$. When it reaches the equilibrium, what is the temperature at $x = 0$?