[Problem 3, pdf] Consider $\begin{cases} x^2\phi''(x)+x\phi'(x)+\lambda\phi(x)=0, & 1<x<e, \\ \phi(1)=\phi(e)=0. \end{cases}$

a) Find all the eigenvalues and eigenfunctions.

b) Construct a general solution to

$$x^2 u_{xx}(x,t)+xu_x(x,t)=u_t(x,t), \quad 1<x<e,\ t>0,$$

given that $u(1,t)=0=u(e,t)$ for $t\geq 0$.

Sol:

- $\lambda=\alpha^2>0$

Try with x^p: $x^2p(p-1)x^{p-2}+xpx^{p-1}+\lambda x^p=0 \Rightarrow$

$$\Rightarrow p^2+\alpha^2=0 \Rightarrow p=\pm|\alpha|i$$

Thus, $\phi(x)=c_1x^{|\alpha|i}+c_2x^{-|\alpha|i}$ $(c_1,c_2\in\mathbb{C})$.

Notice that $x^{|\alpha|i}=e^{\ln x^{|\alpha|i}}=e^{|\alpha|i\ln x}=\cos(|\alpha|\ln x)+i\sin(|\alpha|\ln x)$.

Doing the same with $x^{-|\alpha|i}$, we find that

$$\phi(x)=(a_1+ib_1)(\cos(|\alpha|\ln x)+i\sin(|\alpha|\ln x))+$$
$$+(a_2+ib_2)(\cos(|\alpha|\ln x)-i\sin(|\alpha|\ln x)), \text{ where}$$

we are writing $c_1=a_1+ib_1$, $c_2=a_2+ib_2$, $a_1,a_2,b_1,b_2\in\mathbb{R}$.

Since we are only interested in real solutions, we get

$$\phi(x) = (a_1 + a_2)\cos(|\alpha| \ln x) + (b_2 - b_1)\sin(|\alpha| \ln x).$$

Renaming the constant,

$$\phi(x) = c_3 \cos(|\alpha| \ln x) + c_4 \sin(|\alpha| \ln x).$$

[<u>Remark</u>: We don't need to show all these steps everytime.]

B.C.: $\phi(1) = 0 = c_3$

$\phi(e) = c_4 \sin(|\alpha| \ln e) = 0 \rightsquigarrow |\alpha| = n\pi,\ n = 1, 2, \ldots$

Thus, $\lambda = (n\pi)^2$, $n = 1, 2, \ldots$ are eigenvalues,

with $\phi(x) = c \cdot \sin(n\pi \ln x)$ eigenfunctions.

- $\underline{\lambda = 0}$: $\frac{d}{dx}(x\phi'(x)) = 0 \rightsquigarrow x\phi'(x) = c_1 \rightsquigarrow$

$\rightsquigarrow \phi(x) = c_1 \ln x + c_2$

B.C.:
$$\left.\begin{aligned} \phi(1) &= c_2 = 0 \\ \phi(e) &= c_1 = 0 \end{aligned}\right\} \rightsquigarrow \lambda = 0 \text{ is not an eigenvalue.}$$

• $\underline{\lambda = -\alpha^2 < 0}$: Try with $x^p \to p = \pm|\alpha|$.

$$\phi(x) = c_1 x^{|\alpha|} + c_2 x^{-|\alpha|}.$$

$\underline{BC}$: $$\left.\begin{aligned} \phi(1) &= c_1 + c_2 = 0 \\ \phi(e) &= c_1 e^{|\alpha|} + c_2 e^{-|\alpha|} = 0 \end{aligned}\right\} \to c_1 = c_2 = 0.$$

Therefore, there are not negative eigenvalues.

b) $x^2 u_{xx}(x,t) + x u_x(x,t) = u_t(x,t)$, $1 < x < e$, $t > 0$,

$u(1,t) = 0 = u(e,t)$, $t \geq 0$.

$\underline{Sol}$) $u(x,t) = h(t)\phi(x) \to$

$$\to x^2 h(t)\phi''(x) + x\, h(t)\phi'(x) = h'(t)\phi(x) \to$$

$$\to x^2 \frac{\phi''(x)}{\phi(x)} + x\frac{\phi'(x)}{\phi(x)} = \frac{h'(t)}{h(t)} = -\lambda \to$$

$\to$ ① $h'(t) = -\lambda h(t)$

② $x^2 \phi''(x) + x\phi'(x) + \lambda\phi(x) = 0$

$\underline{BC}$. $u(1,t) = 0 \to \phi(1) = 0$

$u(e,t) = 0 \to \phi(e) = 0$.

Thus, from part a),

$$\lambda = (n\pi)^2, \; n = 1, 2, \dots \quad \text{with} \quad \Phi(x) = c \sin(n\pi \ln x).$$

- ① $h(t) = e^{-t}$.

- Product solution: $u(x,t) = c \sin(n\pi \ln x)\, e^{-t}$.

- Superposition principle:

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(n\pi \ln x)\, e^{-t}.$$

The B_n are undetermined since we don't have initial condition.

• <u>Problem 1:</u> Let $u(x,t)$ be the temperature in a rod, that satisfies

$$\begin{cases} u_t = 3u_{xx} + x^3, \quad 0 < x < 4, \ t > 0, \\ u_x(0,t) = 1 \\ u_x(4,t) = 5 \\ u(x,0) = x. \end{cases}$$

The total energy is defined by $E(t) = \int_0^4 u(x,t)\,dx$.

Find $E(t)$.

<u>Sol:</u> Integrate the equation from $x=0$ to $x=4$:

$$\int_0^4 u_t(x,t)\,dx = 3\int_0^4 u_{xx}(x,t)\,dx + \int_0^4 x^3\,dx \Rightarrow$$

$$\Rightarrow \frac{d}{dt}\int_0^4 u(x,t)\,dx = 3\,u_x(x,t)\Big]_0^4 + \frac{x^4}{4}\Big]_0^4 \Rightarrow$$

$$\Rightarrow \frac{dE}{dt}(t) = 3(5-1) + 4^3 = 12 + 56 = 68 \Rightarrow$$

$$\Rightarrow E(t) = 68t + c.$$

Using the initial condition, we find c:

$$E(0) = c = \int_0^4 u(x,0)\,dx = \int_0^4 x\,dx = \frac{x^2}{2}\Big]_0^4 = 8.$$

Therefore,

$$E(t) = 68t + 8 \,/\!/.$$

• <u>Problem 4</u>: Find $u(x,t)$: $u_t - u = 7x$ with $u(x,0)=0$.

<u>Sol:</u> Notice that e^{-t} is an integrating factor:

$$\frac{\partial}{\partial t}\left(e^{-t}u(x,t)\right) = 7xe^{-t} \Rightarrow e^{-t}u(x,t) = -7xe^{-t} + c(x)$$

we are integrating in t!

$$\left(\underline{\text{Check}}: \quad \frac{\partial}{\partial t}\left[e^{-t}u = -7xe^{-t} + c(x)\right] \rightsquigarrow \frac{\partial}{\partial t}\left(e^{-t}u\right) = 7xe^{-t} \,/\!/ \right)$$

Thus,

$$u(x,t) = -7x + c(x)e^{t}.$$

Then, $u(x,0) = -7x + c(x) = 0 \Leftrightarrow c(x) = 7x$.

In conclusion, $u(x,t) = 7x\left(e^{t}-1\right) /\!/.$

• <u>Problem 5:</u>

Solution: We can simply use linearity.

Let $w = 3u + 5v$. We can check that

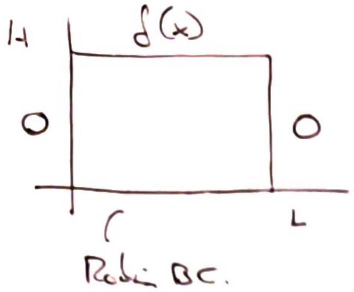
$$w_t = 3u_t + 5v_t = 3\Delta u + 2\cdot 3u + 3e^t \sin(x+2y) +$$
$$+ 5\Delta v + 2\cdot 5v =$$
$$= \Delta(3u+5v) + 2(3u+5v) + 3e^t \sin(x+2y) \Rightarrow$$

$$\Rightarrow w_t = \Delta w + 2w + 3e^t \sin(x+2y) \quad \checkmark.$$

Also, $w(x,y,t) = 3\underbrace{\cancel{u(x,y,t)}}_{=0 \text{ on } \partial\Omega} + 5v(x,y,t) = 5x^2y$ on $\partial\Omega$,

and $w(x,y,0) = 3\cancel{u(x,y,0)} + 5v(x,y,0) = 5\cos(2x)$. //.

• Problem 9: (Solution)



$u(0,y) = 0 = u(L,y)\,, \quad 0 \le y \le H,$

$u(x,0) - u_y(x,0) = 0\,, \quad 0 \le x \le L,$

$u(x,H) = f(x).$

$u(x,y) = F(x)G(y)$

BC: $u(0,y) = 0 \Rightarrow F(0) = 0$

$u(L,y) = 0 \Rightarrow F(L) = 0$ $\Big\} \Rightarrow$ BVP for F.

$u(x,0) - u_y(x,0) = 0 \Rightarrow F(x)G(0) - F(x)G'(0) = 0 \Rightarrow$

$\Rightarrow G(0) - G'(0) = 0.$

$$\frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)} = -\lambda.$$

① $F''(x) + \lambda F(x) = 0$

$F(0) = 0 = F(L)$ $\Big\} \Rightarrow F(x) = \sin\left(\frac{n\pi x}{L}\right),\ n = 1, 2, \dots$

$\left(\lambda = \left(\frac{n\pi}{L}\right)^2\right).$

② $G''(y) - \left(\frac{n\pi}{L}\right)^2 G(y) = 0$

$G(0) - G'(0) = 0$ $\Big\} \Rightarrow G(y) = c_1 \cosh\left(\frac{n\pi y}{L}\right) + c_2 \sinh\left(\frac{n\pi y}{L}\right).$

$\Rightarrow G'(y) = c_1 \frac{n\pi}{L} \sinh\left(\frac{n\pi y}{L}\right) + c_2 \frac{n\pi}{L} \cosh\left(\frac{n\pi y}{L}\right).$

$$G(0) - G'(0) = 0 = c_1 - c_2 \frac{n\pi}{L} \Rightarrow c_2 = \frac{L}{n\pi} c_1 .$$

So,

$$G(y) = c_1 \left(\cosh\left(\frac{n\pi y}{L}\right) + \frac{L}{n\pi} \sinh\left(\frac{n\pi y}{L}\right)\right).$$

- Product sol:

$$u(x,y) = c \sin\left(\frac{n\pi x}{L}\right) \left(\cosh\left(\frac{n\pi y}{L}\right) + \frac{L}{n\pi} \sinh\left(\frac{n\pi y}{L}\right)\right).$$

- Superp. principle:

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\left(\cosh\left(\frac{n\pi y}{L}\right) + \frac{L}{n\pi} \sinh\left(\frac{n\pi y}{L}\right)\right).$$

- B_n? Nonhomogeneous BC:

$$u(x,H) = f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\left(\cosh\left(\frac{n\pi H}{L}\right) + \frac{L}{n\pi} \sinh\left(\frac{n\pi H}{L}\right)\right).$$

Orthogonality of $\sin\left(\frac{n\pi x}{L}\right)$ on $[0,L]$:

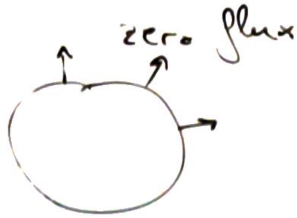
$$\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx = \frac{L}{2} B_n \left(\cosh\left(\frac{n\pi H}{L}\right) + \frac{L}{n\pi} \sinh\left(\frac{n\pi H}{L}\right)\right) \Rightarrow$$

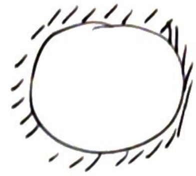
$$\Rightarrow B_n = \left(\cosh\left(\frac{n\pi H}{L}\right) + \frac{L}{n\pi} \sinh\left(\frac{n\pi H}{L}\right)\right)^{-1} \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \; //.$$

• Problem 10: (solution)

$$\left.\begin{aligned} u_t &= \Delta u \quad \text{in } D \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial D \end{aligned}\right\}$$

$$u(x,y,0) = f(x,y)$$





Insulated disk.

Notice that for a disk,

$\frac{\partial u}{\partial n}(r,\theta) = u_r(r,\theta)$, since the normal vector points in the radial direction.

• Equilibrium solution $\Rightarrow u_t = 0$, $u(x,y,t) = u(x,y)$ at equil.

So we have to solve the Laplace equation $\begin{cases} \Delta u = 0 \\ u_r(1,\theta) = 0 \end{cases}$

(with the extra information of the initial data, so we can use the conservation of energy at the end).

$$u(r,\theta) = F(r)G(\theta)$$

$$\hookrightarrow \frac{1}{r} G(\theta) \frac{d}{dr}(r F'(r)) + \frac{1}{r^2} F(r) G''(\theta) = 0$$

B.C.: $u_r(1,\theta) = 0 \rightarrow F'(1) = 0$

$$|u(0,\theta)| < \infty$$

$$\left.\begin{aligned} u(r,\pi) &= u(r,-\pi) \\ u_\theta(r,\pi) &= u_\theta(r,-\pi) \end{aligned}\right\} \rightarrow \begin{aligned} G(-\pi) &= G(\pi) \\ G'(-\pi) &= G'(\pi) \end{aligned}$$

$$\frac{r}{F(r)}\frac{d}{dr}\left(r F'(r)\right) = -\frac{G''(\theta)}{G(\theta)} = \lambda$$

① $G''(\theta) + \lambda G(\theta) = 0$
periodic BC $\quad \Rightarrow \quad \lambda = n^2 \quad , n = 0, 1, \ldots$

$$G(\theta) = c_1 \cos(n\theta) + c_2 \sin(n\theta) .$$

② $r \frac{d}{dr}\left(r F'(r)\right) = \lambda F(r)$

$\hookrightarrow \lambda = 0 \to r F'(r) = c_3 \Rightarrow F(r) = c_3 \ln(r) + c_4 .$

Since $|u(0,\theta)| < \infty \Rightarrow c_3 = 0$ //.

• $\lambda = n^2 \quad , n = 1, 2, \ldots$

$$r^2 F''(r) + r F'(r) - n^2 F(r) = 0$$

Try with $F(r) = r^p \leadsto p = \pm n$.

$$F(r) = c_5 r^n + c_6 r^{-n} \underset{\substack{\uparrow \\ |u(0,\theta)| < \infty}}{\Longrightarrow} F(r) = c_5 r^n .$$

In summary,

$$F(r) = c\, r^n \quad , n = 0, 1, \ldots$$

- Product sol.: $u(r,\theta) = r^n \left(c_1 \cos(n\theta) + c_2 \sin(n\theta)\right)$

- Superp. principle:

$$u(r,\theta) = \sum_{n=0}^{\infty} A^n r^n \cos(n\theta) + \sum_{n=1}^{\infty} B^n r^n \sin(n\theta).$$

$$u_r(1,\theta) = 0 = \sum_{n=0}^{\infty} A^n n \cos(n\theta) + \sum_{n=1}^{\infty} B^n n \sin(n\theta) \Rightarrow$$

$$\Rightarrow \begin{matrix} A^n n = 0 \\ B^n n = 0 \end{matrix} \Rightarrow A_n = B_n = 0 \text{ for } n \geq 1. \qquad (*)$$

So $u(r,\theta) = A_0$. To find A_0, we use the I.C.:

$$\iint_D u_t \, dS = \iint_D \Delta u \, dS = \int_{\partial D} \frac{\partial u}{\partial n} \, dl = 0 \Rightarrow$$

$$\Rightarrow \iint_D u \, dS = \text{constant} \Rightarrow \left\| A_0 = \frac{1}{\pi} \iint_D f \, dS = u \right\|$$

area of unit disk

equilibrium sol.

(*) <u>Remark</u>: We knew this without doing any computation! The disk is insulated, the initial temperature distribution diffuses to a constant value in such a way that the total energy is the same. This happens when the final constant temperature is the average of the initial ones.