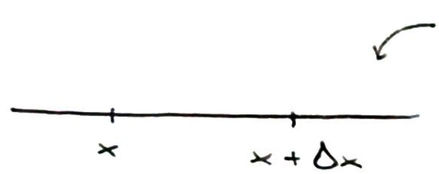
MATH 241 LECTURE 10 : The Wave Equation.

• Introduction: Approximate Physical Derivation.

Every phenomena of oscillatory character is related to the so-called wave equation: waves in the sea, movement of a string or a membrane of a drum, the propagation of sound, etc.

Here, we will obtain a <u>linear</u> approximation. We show it for the movement of a string:

thin, stretched string, with density $\rho(x)$.



• Hypothesis: 1) The string is perfectly flexible, that is, it offers no resistance to bending. In practice, this means that the string only exerts force in the tangential direction → <u>Tension force</u>.

2) Small slopes

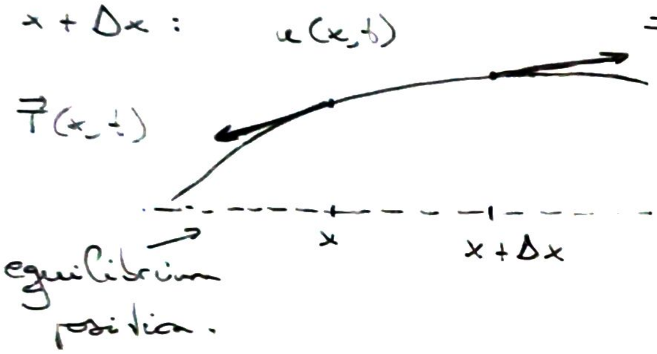
3) Horizontal displacements are negligible.

Hypothesis 1)–3) can be removed for more realistic models, but it is not easy and the resulting equations are nonlinear (very hard).

↳ But of course necessary in research. (waves in the sea might be wild! very nonlinear in nature).

→ Nevertheless, the hypothesis are very reasonable for the study of strings in musical instruments, for example.

• Consider a small arbitrary segment between x and $x + \Delta x$:

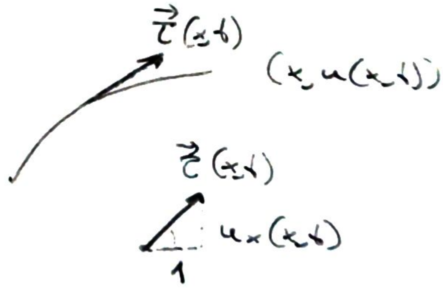


$\vec{T}$ = tension force.

$u(x,t) \equiv$ vertical displacement from equilibrium.

Notice that the graph of the curve is parametrized by $(x, u(x,t))$, so the unit tangent vector at a point x is

$$\vec{\tau}(x,t) = \frac{(1, u_x(x,t))}{\sqrt{1 + (u_x(x,t))^2}}.$$



Notice that the slope at x is given by $u_x(x,t)$, which by hypothesis 2) is small. So we can linearize (or Taylor expand):

$$\sqrt{1+u_x^2} \approx 1 \quad \text{for } |u_x| \ll 1.$$

$$\hookrightarrow \vec{\tau}(x,t) = (1, u_x(x,t)).$$

- Then, we can write the tension force in terms of its magnitude:

$$\vec{T}(x,t) = T(x,t)\,\vec{\tau}(x,t) = T(x,t)\begin{pmatrix}1\\ u_x(x,t)\end{pmatrix}.$$

- Finally, let's apply Newton's Second Law for the segment of the string between x and $x+\Delta x$:

$$\underbrace{\rho(x)\,\Delta x}_{(*)\ \text{mass}}\ \underbrace{u_{tt}(x,t)}_{\text{vertical acceleration}} = T(x+\Delta x,t)\,u_x(x+\Delta x,t) - T(x,t)\,u_x(x,t) + \underbrace{\rho(x)\,\Delta x}_{\text{mass}}\ \underbrace{Q(x,t)}_{\text{body forces such as gravity}}$$

(vertical) tension at both ends

(*) The length is approx. Δx because $|u_x| \ll 1$.

small slope.

Divide by Δx and let $\Delta x \to 0$ to find that

$$\rho(x)\, u_{tt}(x,t) = \frac{\partial}{\partial x}\left(T(x,t)\, u_x(x,t)\right) + \rho(x)\, Q(x,t)$$

- The one-dimensional wave equation is

$$\boxed{u_{tt} = c^2 u_{xx}}$$

which comes from the above one when $\rho(x) = \rho$, $T(x,t) = T$, $Q(x,t) = 0$, and $c^2 = \frac{T}{\rho}$.

- <u>Boundary Conditions</u>

We can consider many BCs. The easiest ones to interpret are Dirichlet zero BCs,

$$\left.\begin{aligned} u(0,t) &= 0 \\ u(L,t) &= 0 \end{aligned}\right\}$$ ends are fixed with zero displacement.

Non-homogeneous Dirichlet BCs, like $u(0,t) = f(t)$, mean that the movement of that end is externally prescribed.

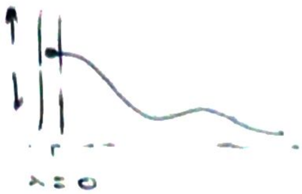
→ Neumann and Robin BC have also a physical interpretation, but less obvious (see R. Haberman pages

134 to 136 if interested in the physical meaning).

Basically, a homogeneous Neumann BC,

$u_x(0,t)=0$,

means that that end of the string can move freely (but only vertically).



Indeed, remember that the tension force at a point x was $T u_x(x,t)$. So zero Neumann BC means there is no force acting there).

→ Mathematically, the method of separation of variables will work for all these BCs, as in the heat equation.

- <u>Solution</u>: Separation of Variables

<u>Example 1</u>: String with fixed ends

PDE $\quad u_{tt} = c^2 u_{xx}$, $0 < x < L$, $t > 0$,

BC $\quad u(0,t) = 0$, $u(L,t) = 0$, $t > 0$ ---- (Homogeneous Dirichlet BC)

IC $\quad u(x,0) = f(x)$, $0 < x < L$

$u_t(x,0) = g(x)$ ← initial velocity

We need two initial conditions because the equation has a second order derivative in time.

(Think of Newton's law: to know the position at later times, we need to know now the position <u>and</u> the velocity.)

Solution: Let $u(x,t) = \phi(x)\, G(t)$

BC: $u(0,t) = \phi(0)\, G(t) = 0 \quad \forall t > 0$ $\Rightarrow$ $G(t) = 0$ for all t (trivial case) or $\underline{\underline{\phi(0) = 0}}$.

$u(L,t) = \phi(L)\, G(t) = 0 \quad \forall t > 0$ $\Rightarrow$ $G(t) = 0 \ \forall t > 0$ (trivial case) or $\underline{\underline{\phi(L) = 0}}$

we cannot use the initial conditions yet

PDE: $\Phi(x)G''(t) = c^2\Phi''(x)G(t) \Rightarrow$

$$\Rightarrow \frac{1}{c^2}\frac{G''(t)}{G(t)} = \frac{\Phi''(x)}{\Phi(x)} = -\lambda.$$

① $\left.\begin{array}{l}\Phi''(x) + \lambda\Phi(x) = 0 \\ \Phi(0) = 0 = \Phi(L)\end{array}\right\} \to \begin{array}{l}\lambda = \left(\frac{n\pi}{L}\right)^2, \ n = 1, 2, \ldots \\ \Phi(x) = \sin\left(\frac{n\pi x}{L}\right)\end{array}$

② $G''(t) = -c^2\left(\frac{n\pi}{L}\right)^2 G(t)$

$\to G(t) = c_1\cos\left(\frac{n\pi ct}{L}\right) + c_2\sin\left(\frac{n\pi ct}{L}\right).$

- Thus, the product solutions are

$$u(x,t) = \left(c_1\cos\left(\frac{n\pi ct}{L}\right) + c_2\sin\left(\frac{n\pi ct}{L}\right)\right)\sin\left(\frac{n\pi x}{L}\right),$$

and by the superposition principle

$$u(x,t) = \sum_{n=1}^{\infty}\left(c_n\cos\left(\frac{n\pi ct}{L}\right) + d_n\sin\left(\frac{n\pi ct}{L}\right)\right)\sin\left(\frac{n\pi x}{L}\right).$$

- Finally, we impose the initial conditions:

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)$$

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} d_n \frac{n\pi c}{L} \sin\left(\frac{n\pi x}{L}\right)$$

$\rightarrow$ (Fourier Sine series)

$$\Rightarrow \quad c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$d_n \frac{n\pi c}{L} = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx. \quad //$$

<u>Example 2:</u> Damped wave equation $(\alpha > 0)$

PDE $\quad u_{tt} = c^2 u_{xx} - \alpha u_t, \quad 0 < x < L, \; t > 0$

B.C. $\quad u_x(0,t) = 0, \quad u(L,t) = 0, \quad t > 0$

IC: $\quad u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 < x < L$

Find the solution under the assumption that $\alpha^2 < c^2 \frac{\pi^2}{L^2}$.

Solution: $u(x,t) = \phi(x)\, G(t)$

BC:

$$u_x(0,t) = \phi'(0) G(t) = 0 \Rightarrow \phi'(0) = 0$$

$$u(L,t) = \phi(L) G(t) = 0 \Rightarrow \phi(L) = 0$$

PDE:

$$\phi(x) G''(t) = c^2 \phi''(x) G(t) - \alpha \phi(x) G'(t) \Rightarrow$$

$$\Rightarrow \frac{1}{c^2}\frac{G''(t)}{G(t)} + \frac{\alpha}{c^2}\frac{G'(t)}{G(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda.$$

① $\left.\begin{array}{l} \phi''(x) + \lambda\phi(x) \\ \phi'(0) = 0 = \phi(L) \end{array}\right\}$

$\underline{\lambda > 0}$: $\phi(x) = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$

$$\phi'(0) = c_2\sqrt{\lambda} = 0 \Rightarrow c_2 = 0$$

$$\phi(L) = c_1 \cos(\sqrt{\lambda}\, L) = 0 \Rightarrow \sqrt{\lambda}\, L = \frac{\pi}{2} + n\pi \Rightarrow$$

$$\Rightarrow \lambda = \left((2n+1)\frac{\pi}{2L}\right)^2, \quad n = 0, 1, \ldots$$

$$\phi(x) = c_1 \cos(\sqrt{\lambda}\, x)$$

$\underline{\lambda = 0}$: $\phi(x) = c_3 x + c_4$

$$\left.\begin{array}{l} \phi'(0) = c_3 = 0 \\ \phi(L) = c_4 = 0 \end{array}\right| \Rightarrow \phi \equiv 0,$$

$\underline{\lambda < 0}$: No eigenvalues (... done before..)

(i) $G''(t) + \alpha G'(t) + c^2\left(\frac{2n+1}{2}\right)^2\left(\frac{\pi}{L}\right)^2 G(t) = 0$, $n = 0, 1, \ldots$

Characteristic polynomial:

$$r^2 + \alpha r + c^2\left(\frac{2n+1}{2}\right)^2\left(\frac{\pi}{L}\right)^2 = 0 \Rightarrow$$

$$\Rightarrow r = \frac{-\alpha \pm \sqrt{\alpha^2 - 4c^2\frac{(2n+1)^2}{4}\frac{\pi^2}{L^2}}}{2} = \frac{-\alpha \pm \sqrt{\alpha^2 - c^2\frac{\pi^2}{L^2}(2n+1)^2}}{2}.$$

Notice that for $\alpha^2 < c^2\frac{\pi^2}{L^2}$ (assumption given in the statement), for any $n \geq 0$ we have that $\alpha^2 - c^2\frac{\pi^2}{L^2}(2n+1)^2 < 0$.

Therefore,

$$r = \frac{-\alpha}{2} \pm \frac{i}{2}\sqrt{(2n+1)^2\frac{\pi^2}{L^2}c^2 - \alpha^2} = \frac{-\alpha}{2} \pm i\,\omega_n,$$

with $\omega_n = \frac{1}{2}\sqrt{(2n+1)^2\frac{\pi^2}{L^2}c^2 - \alpha^2}$.

Then,

$$G(t) = e^{-\frac{\alpha}{2}t}\left(c_5\cos(\omega_n t) + c_6\sin(\omega_n t)\right).$$

- Superposition principle:

$$u(x,t) = \sum_{n=0}^{\infty} e^{-\frac{\alpha}{2}t}\left(A_n\cos(\omega_n t) + B_n\sin(\omega_n t)\right)\cos\left(\frac{2n+1}{2}\frac{\pi x}{L}\right)$$

• Initial conditions:

$$u(x,0) = f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{2n+1}{2}\frac{\pi x}{L}\right). \quad (1)$$

and

$$u_t(x,t) = \sum_{n=0}^{\infty} \frac{-\alpha}{2} e^{-\frac{\alpha}{2}t}\left(A_n \cos(\omega_n t) + B_n \sin(\omega_n t)\right) \cos\left(\frac{(2n+1)\pi x}{2L}\right) +$$

$$+ \sum_{n=0}^{\infty} e^{-\frac{\alpha}{2}t}\left(-A_n \omega_n \sin(\omega_n t) + B_n \omega_n \cos(\omega_n t)\right) \cos\left(\frac{(2n+1)\pi x}{2L}\right)$$

$$\hookrightarrow u_t(x,0) = g(x) = \sum_{n=0}^{\infty} \frac{-\alpha}{2} A_n \cos\left(\frac{(2n+1)\pi x}{2L}\right) +$$

$$+ \sum_{n=0}^{\infty} B_n \omega_n \cos\left(\frac{(2n+1)\pi x}{2L}\right) =$$

$$= \sum_{n=0}^{\infty} \left(\frac{-\alpha}{2} A_n + B_n \omega_n\right) \cos\left(\frac{(2n+1)\pi x}{2L}\right) \quad (2)$$

Finally, $(1) \Rightarrow A_n = \frac{2}{L}\int_0^L f(x) \cos\left(\frac{2n+1}{2}\frac{\pi x}{2}\right) dx \Rightarrow A_n$ //

$(2) \Rightarrow \frac{-\alpha}{2} A_n + B_n \omega_n = \frac{2}{L}\int_0^L g(x) \cos\left(\frac{2n+1}{2}\frac{\pi x}{2}\right) dx$

$\hookrightarrow B_n$ //.