MATH 241 LECTURE 11 : Sturm-Liouville Eigenvalue Problems.

[Continuation of the Wave Equation].

Exercise: Damped wave equation

Solve $u_{tt} = u_{xx} - 3u_t \quad , \; 0 < x < \pi , \; t > 0$

BC $u(0,t) = 0 = u(\pi,t) \quad , \; t > 0$

IC $u(x,0) = f(x)$
$u_t(x,0) = g(x)$, $0 < x < \pi$

Solution: $u(x,t) = \phi(x) G(t)$

BC: $u(0,t) = \phi(0)G(t) = 0 \rightarrow \phi(0) = 0$

$u(\pi,t) = \phi(\pi) G(t) = 0 \rightarrow \phi(\pi) = 0$

PDE:
$$\frac{G''(t)}{G(t)} = \frac{\phi''(x)}{\phi(x)} - 3\,\frac{G'(t)}{G(t)} \rightarrow$$

$$\rightarrow \frac{G''(t)}{G(t)} + 3\,\frac{G'(t)}{G(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda$$

① BVP $\phi''(x) + \lambda\phi(x) = 0$, $\phi(0) = \phi(\pi) = 0$ $\rightarrow$ $\lambda = n^2, \; n = 1, 2, \ldots$ $\phi(x) = \sin(nx)$

c) $G''(t) + 3G'(t) + n^2 G(t) = 0 \quad (n = 1, 2, \ldots)$

Charact. polynomial $\to r^2 + 3r + n^2 = 0 \to$

$$\to r = \frac{-3 \pm \sqrt{9 - 4n^2}}{2}$$

we distinguish two cases here:

1) $n = 1 \to r = \frac{-3}{2} \pm \frac{\sqrt{5}}{2} \to r_1 = \frac{-3+\sqrt{5}}{2}, \; r_2 = \frac{-3-\sqrt{5}}{2}$

In this case, $G(t) = c_1 e^{\frac{-3+\sqrt{5}}{2}t} + c_2 e^{\frac{-3-\sqrt{5}}{2}}$.

2) $n \geq 2 \to$ Then, $9 - 4n^2$ is negative, so

$r = \frac{-3}{2} \pm i \frac{\sqrt{4n^2-9}}{2}$. Therefore,

$$G(t) = e^{-\frac{3}{2}t}\left(c_1 \cos\left(\frac{\sqrt{4n^2-9}}{2}t\right) + c_2 \sin\left(\frac{\sqrt{4n^2-9}}{2}t\right)\right).$$

• Finally, we apply the superposition principle:

$$u(x,t) = \left(A_1 e^{\frac{-3+\sqrt{5}}{2}t} + B_1 e^{\frac{-3-\sqrt{5}}{2}t}\right)\sin(x) +$$

$$+ \sum_{n=2}^{\infty} e^{-\frac{3}{2}t}\left(A_n \cos\left(\frac{\sqrt{4n^2-9}}{2}t\right) + B_n \sin\left(\frac{\sqrt{4n^2-9}}{2}t\right)\right)\sin(nx).$$

and use the initial conditions to find A_n, B_n:

$$u(x,0) = f(x) = (A_1 + B_1)\sin(x) + \sum_{n=2}^{\infty} A_n \sin(nx),$$

$$u_t(x,0) = g(x) = \left(\frac{-3+\sqrt{5}}{2}A_1 + \frac{-3-\sqrt{5}}{2}B_1\right)\sin(x) + \sum_{n=2}^{\infty} \frac{-3}{2}\frac{\sqrt{4n^2-9}}{2} B_n \sin(nx)$$

$\Rightarrow$

$$A_1 + B_1 = \frac{2}{\pi}\int_0^\pi f(x)\sin(x)\,dx, \quad A_n = \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\,dx, \quad n \geq 2.$$

$$\underbrace{\frac{-3+\sqrt{5}}{2}A_1 - \frac{3+\sqrt{5}}{2}B_1 = \frac{2}{\pi}\int_0^\pi g(x)\sin(x)\,dx}_{\text{system of two variables } (A_1, B_1) \text{ with two equations}}, \quad \frac{-3}{2}\frac{\sqrt{4n^2-9}}{2}B_n = \frac{2}{\pi}\int_0^\pi g(x)\sin(nx)\,dx, \quad n \geq 2.$$

- **Wave equation and energy.**

In the physical derivation of the wave equation, $u_{tt} = c^2 u_{xx}$, the only force considered was the tension force, given by Tu_x. This force is non-dissipative, and therefore the "energy" should be conserved in time.

Let's check that this is indeed true.

Define the total energy by

$$E(t) = \underbrace{\frac{1}{2}\rho\int_0^L (u_t(x,t))^2\,dx}_{\text{kinetic energy}} + \underbrace{\frac{T}{2}\int_0^L (u_x(x,t))^2\,dx}_{\text{potential energy}}.$$

Then,

$$E'(t) = \rho\int_0^L u_t(x,t)\,u_{tt}(x,t)\,dx + T\int_0^L u_x(x,t)\,u_{xt}(x,t)\,dx.$$

<u>Integrate by parts</u> in the second term,

$$\left\{\begin{array}{ll} v = u_x \longrightarrow dv = u_{xx}\,dx & \\ dw = u_{xt}\,dx \longrightarrow w = u_t & , \quad \int v\,dw = vw - \int w\,dv \end{array}\right\}$$

$$E'(t) = \rho\int_0^L u_t(x,t)\,u_{tt}(x,t)\,dx - T\int_0^L u_t(x,t)\,u_{xx}(x,t)\,dx +$$

$$+ \left. u_x(x,t)\,u_t(x,t)\right]_0^L =$$

$$= \rho\int_0^L u_t(x,t)\Big(\underbrace{u_{tt}(x,t) - c^2 u_{xx}(x,t)}_{=0}\Big)\,dx + \left. u_x(x,t)\,u_t(x,t)\right]_0^L.$$

That is,

$$E'(t) = \left. u_x(x,t)\,u_t(x,t)\right]_0^L$$

Recall that the force was Tu_x, so

$u_x(x,t)$ at $x=0, x=L \rightarrow$ force externally applied at $x=0$, $x=L$.

$u_t(x,t)$ at $x=0, x=L \rightarrow$ velocity externally imposed at the end.

Therefore, we can see if that the ends move freely (i.e., $u_x(x,t)=0$ for $x=0$ or $x=L$) or if they are fixed (and thus they have zero velocity), then

$$E'(t) = 0 \Rightarrow E(t) = E(0)$$

(Dirichlet or homogeneous Neumann B.C.).

Chapter 5: Sturm-Liouville Eigenvalue Problems.

So far in the course, we have solve a few PDE such as

$u_t = k u_{xx}$ Heat equation (1d)

$u_{xx} + u_{yy} = 0$ (2d) Laplace equation [constants coefficients case]

$u_{tt} = c^2 u_{xx}$ Wave equation (1d)

by applying separation of variables:

→ Write u as a product of one-variable functions.

→ Split the PDE into two ODEs.

→ One of the ODEs together with two homogeneous BCs gave rise to an "eigenvalue problem". So far, it has always been

$$\left.\begin{array}{l} \phi''(x) + \lambda \phi(x) = 0 \\ + \text{ two (linear) homogeneous BCs.} \end{array}\right\} [*]$$

→ With the values of λ previously found, solve the other ODE.

→ After constructing a general solution by the superposition principle (that solves the PDE and the homogeneous BCs), we use the initial data (or the nonhomogeneous BC) to find the constants of the general solution. ← We crucially used the orthogonality property of the eigenfunctions. (sines, cosines)

[*] So far, the eigenvalue problem has always given nonnegative eigenvalues $\lambda \geq 0$, and we've been able to explicitly compute all λ, ϕ.

• We will see now that, in general, we cannot solve the eigenvalue problem for most PDEs, but all the previous properties will still be true.

• <u>Example / motivation:</u> Heat equation with non-constant coeff.

PDE $$c(x)\rho(x)u_t(x,t) = \frac{\partial}{\partial x}\left(K(x)u_x(x,t)\right) + \alpha(x)u(x).$$

↳ Homogeneous and linear ⇝ we can try to use separation of variables.

Let $u(x,t) = \phi(x)G(t) \rightarrow$

$$\rightarrow c(x)\rho(x)\phi(x)G'(t) = \frac{\partial}{\partial x}\left(K(x)\phi'(x)G(t)\right) + \alpha(x)\phi(x)G(t)$$

$$c(x)\rho(x)\Phi(x)G'(x) = G(t)\frac{d}{dx}\left(k(x)\Phi'(x)\right) + \alpha(x)\Phi(x)G(t) \Rightarrow$$

$$\frac{G'(t)}{G(t)} = \frac{1}{c(x)\rho(x)\Phi(x)}\frac{d}{dx}\left(k(x)\Phi'(x)\right) + \frac{\alpha(x)}{c(x)\rho(x)} = -\lambda \Rightarrow$$

$$\Rightarrow \text{①}\ \frac{d}{dx}\left(k(x)\frac{d\Phi}{dx}\right) + \alpha(x)\Phi(x) + \lambda c(x)\rho(x)\Phi(x) = 0$$

$$\text{②}\ G'(t) = -\lambda G(t).$$

Say that we had fixed temperature BCs $u(0,t) = 0 = u(L,t)$, then,

$$\text{BVP:}\ \left.\begin{aligned} &\frac{d}{dx}\left(k(x)\frac{d\Phi}{dx}(x)\right) + \alpha(x)\Phi(x) + \lambda c(x)\rho(x)\Phi(x) = 0 \\ &\Phi(0) = 0 \\ &\Phi(L) = 0 \end{aligned}\right\}$$

↳ Eigenvalue Problem ⇝ We cannot (in general) solve for λ and Φ explicitly (only numerically)

(this should give us the λ's and Φ. Then, we'd continue to solve for $G(t)$).

↓

But we would like to obtain certain qualitative properties; for example, λ should be ≥ 0, otherwise ② is unphysical.

- <u>Def</u>: Sturm-Liouville Differential Equation

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}(x)\right) + g(x)\phi(x) + \lambda\sigma(x)\phi(x) = 0 , \quad a < x < b.$$

→ The previous example $\frac{d}{dx}\left(k(x)\frac{d\phi}{dx}(x)\right) + \alpha(x)\phi(x) + \lambda c(x)\rho(x)\phi(x) = 0$,

was of this form with $p(x) = k(x)$, $g(x) = \alpha(x)$, $\sigma(x) = c(x)\rho(x)$.

The typical case $\phi''(x) + \lambda\phi(x) = 0$ too, with

$p(x) = 1$, $g(x) = 0$, $\sigma(x) = 1$.

- Boundary conditions: summary (<u>Homogeneous</u> BC).

Types considered:

1) Dirichlet : $\phi(x_0) = 0$ (x_0 a boundary point).

2) Neumann : $\phi'(x_0) = 0$

3) Robin : $\alpha_1\phi'(x_0) + \alpha_2\phi(x_0) = 0$

4) Other : periodicity, singularity

$\phi(-L) = \phi(L)$ $\quad |\phi(x_0)| < \infty$

$\phi'(-L) = \phi'(L)$

• **Def.:** $\boxed{\text{Regular}}$ Sturm-Liouville Eigenvalue Problem.

ODE: $$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}(x)\right) + q(x)\phi(x) + \lambda\sigma(x)\phi(x) = 0, \quad a < x < b,$$

+

BC: $$\begin{cases} \alpha_1 \phi(a) + \beta_1 \phi'(a) = 0 \\ \alpha_2 \phi(b) + \beta_2 \phi'(b) = 0 \end{cases} \qquad (\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R})$$

with
- $p(x)$, $q(x)$, $\sigma(x)$ real and continuous on $[a,b]$
- $p(x) > 0$ for all $x \in [a,b]$
- $\sigma(x) > 0$

The end points are included!

Remark:
- All conditions are needed for a problem to be called "regular".
- The BC are homogeneous and Dirichlet, Neumann, or Robin. Periodic or singular BC are NOT included.

Examples: Decide if the following are problems are given in the standard Sturm-Liouville form.
If they are, what are $p(x)$, $q(x)$, $\sigma(x)$?
- Are they Regular Sturm-Liouville Eigenvalue problems?