MATH 425
LECTURE 9

Method of Separation of Variables I.

We are going to solve the heat and wave equation on a bounded interval $[0,L]$. The method of separation of variables works in many other situations: basically, it is well suited to deal with linear, homogeneous PDEs with linear and homogeneous BCs.

<u>Example</u>: Superposition principle for the heat equation with Dirichlet zero BC.

If v, w solve $\left.\begin{aligned} u_t &= k u_{xx} \\ 0 = u(0,t) &= u(L,t) \end{aligned}\right\}$ then $\alpha_1 v + \alpha_2 w$ also solve the problem.

<u>Check</u>: PDE $\to$ linearity

BC. $\to$ $\left.\begin{aligned} \alpha_1 v(0,t) + \alpha_2 w(0,t) &= 0 \\ \alpha_1 v(L,t) + \alpha_2 w(0,t) &= 0 \end{aligned}\right\}$ It is crucial that the BC. was homogeneous. ✓

- We are going to describe the method of separation of variables through examples.

- **Example 1:** Heat Equation with Dirichlet BC.

$$\left.\begin{array}{ll} \text{PDE} & u_t = k u_{xx}, \quad 0 < x < L,\ t > 0 \\ \text{BC} & u(0,t) = 0 = u(L,t). \\ \text{IC} & u(x,0) = f(x) \end{array}\right\}$$

Solution:

1) Let $u(x,t) = \phi(x)\,G(t)$. Substitute into the PDE:

$$\phi(x)\,G'(t) = k\,\phi''(x)\,G(t) \rightarrow$$

$$\rightarrow \underbrace{\frac{1}{k}\frac{G'(t)}{G(t)}}_{\text{function of } t} = \underbrace{\frac{\phi''(x)}{\phi(x)}}_{\text{function of } x} = -\lambda$$

$\hookrightarrow$ separation constant.

2) The PDE becomes two ODEs:

① $\phi''(x) + \lambda\phi(x) = 0$ ② $G'(t) = -\lambda k\,G(t)$.

3) Translate the BC to ϕ, G:

$u(0,t) = \phi(0)\,G(t) = 0$ for all $t \geq 0$ $\rightarrow$ ($G(t) \equiv 0 \rightarrow u \equiv 0$) or $\phi(0) = 0$

$u(L,t) = \phi(L)\,G(t) = 0$ for all $t \geq 0$ $\rightarrow$ ($G(t) \equiv 0 \rightarrow u \equiv 0$) or $\phi(L) = 0$

4) Solve the boundary value problem (BVP): That is, the ODE with two homogeneous BC

$$\left.\begin{aligned} \Phi''(x) + \lambda \Phi(x) &= 0 \\ \Phi(0) = \Phi(L) &= 0 \end{aligned}\right\}$$

Remark: The values of λ for which there exist non-trivial solutions Φ (i.e. $\Phi \not\equiv 0$) are called eigenvalues, and the corresponding Φ eigenfunctions

Four possible cases:

• $\lambda > 0$, • $\lambda = 0$, • $\lambda < 0$, • $\underbrace{\lambda \text{ complex}}$.

we will prove later that this never happens in our problems (under certain conditions).

• $\underline{\lambda > 0}$: Then

$$\Phi(x) = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$$

BC $\quad \Phi(0) = c_1 = 0$

$\Phi(L) = c_2 \sin(\sqrt{\lambda} L) = 0$

$\nearrow c_2 = 0 \to \Phi \equiv 0$ (trivial case)

$\searrow \sin(\sqrt{\lambda} L) = 0 \Leftrightarrow$

$\Leftrightarrow \sqrt{\lambda} L = n\pi, \; n = 1, 2, \ldots$

$\Leftrightarrow \lambda = \left(\frac{n\pi}{L}\right)^2$

So, $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $n \geq 1$, are eigenvalues with

$\Phi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$ their eigenfunctions.

• $\lambda = 0$: $\phi(x) = c_1 x + c_2$

$$\left.\begin{aligned} &\phi(0) = c_2 = 0 \\ &\phi(L) = c_1 L = 0 \Rightarrow c_1 = 0 \end{aligned}\right\} \Rightarrow \phi \equiv 0.$$

So $\lambda = 0$ is not an eigenvalue.

• $\lambda < 0$: $\phi(x) = c_1 \cosh(\sqrt{\lambda}\, x) + c_2 \sinh(\sqrt{\lambda}\, x)$

$$\left.\begin{aligned} &\phi(0) = c_1 = 0 \\ &\phi(L) = c_2 \sinh(\sqrt{\lambda}\, L) = 0 \Rightarrow c_2 = 0 \end{aligned}\right\} \Rightarrow \phi \equiv 0.$$

So there are not negative eigenvalues.

5) With the values of λ found in 4), solve the other ODE:

$$G'(t) = c\, e^{-\left(\frac{n\pi}{L}\right)^2 k t}, \quad n \geq 1.$$

6) Any product solution $u_n(x,t) = c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 k t}$ solves the PDE and the homogeneous BC.

7) Apply the superposition principle:

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 k t}$$ also solves the PDE and the homogeneous BC. (*)

8) Use the initial condition to find c_n:

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right).$$

→ Question: Given any $f(x)$, is it possible to write it as an infinite series of sines ??

↳ For the moment we will assume so.

→ (*) We need certain conditions to guarantee the convergence of the series, and moreover, that the series can be derived term by term.

• Example 2 : Wave Equation with Dirichlet BC.

$$\left.\begin{array}{ll} \text{PDE} & u_{tt} = c^2 u_{xx}, \quad 0 < x < L,\ t > 0 \\ \text{BC:} & u(0,t) = 0 \\ & u(L,t) = 0 \\ \text{IC:} & u(x,0) = f(x) \\ & u_t(x,0) = g(x) \end{array}\right\}$$

Solution: $u(x,t) = \phi(x)\, G(t)$

PDE: $\phi(x)\, G''(t) = c^2 \phi''(x) G(t) \rightarrow \frac{1}{c^2}\frac{G''(t)}{G(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda \rightarrow$

① $\phi''(x) + \lambda\phi(x) = 0$ ② $G''(t) = -\lambda c^2 G(t)$.

BC: $u(0,t) = \phi(0)G(t) = 0 \Rightarrow \phi(0) = 0$

$u(L,t) = \phi(L)\,G(t) = 0 \Rightarrow \phi(L) = 0$.

Notice that $\left.\begin{array}{l} ①\ \phi''(x) + \lambda\phi(x) = 0 \\ \phi(0) = 0 = \phi(L) \end{array}\right\}$ is the same eigenvalue problem as before.

↳ Solutions: $\lambda_n = \left(\frac{n\pi}{L}\right)^2, n \geq 1$

$\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$.

② $G''(t) + \left(\frac{n\pi}{L}\right)^2 c^2 G(t) = 0 \Rightarrow$

$\Rightarrow G_n(t) = A_n \cos\left(\frac{n\pi c t}{L}\right) + B_n \sin\left(\frac{n\pi c t}{L}\right)$

• Therefore, product solutions:

$$u_n(x,t) = \left(A_n \cos\left(\frac{n\pi c t}{L}\right) + B_n \sin\left(\frac{n\pi c t}{L}\right)\right)\sin\left(\frac{n\pi x}{L}\right)$$

• Superp. principle:

$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n \cos\left(\frac{n\pi c t}{L}\right) + B_n \sin\left(\frac{n\pi c t}{L}\right)\right)\sin\left(\frac{n\pi x}{L}\right).$$

- Initial conditions:

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right), \quad \rightsquigarrow A_n$$

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L} \sin\left(\frac{n\pi x}{L}\right) \rightsquigarrow B_n$$

[We believe this is possible for the moment].

- Example 3: Heat Equation with Neumann BC.

$$\left.\begin{aligned} &\text{PDE: } && u_t = k u_{xx}, \quad 0 < x < L, \ t > 0 \\ &\text{BC: } && u_x(0,t) = 0 \\ & && u_x(L,t) = 0 \\ &\text{IC: } && u(x,0) = f(x) \end{aligned}\right\}$$

Solution: $u(x,t) = \phi(x) G(t)$

PDE: $\displaystyle \frac{1}{k} \frac{G'(t)}{G(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda$

BC: $u_x(0,t) = \phi'(0) G(t) = 0 \Rightarrow \phi'(0) = 0$

$u_x(L,t) = \phi'(L) G(t) = 0 \Rightarrow \phi'(L) = 0$

BVP: $\left.\begin{aligned} &\phi''(x) + \lambda \phi(x) = 0 \\ &\phi'(0) = 0 = \phi'(L) \end{aligned}\right\}$ we need to consider $\lambda > 0$, $\lambda = 0$, $\lambda < 0$ (λ complex).

• $\underline{\lambda > 0}$: $\phi(x) = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x)$

$$\phi'(x) = -c_1 \sqrt{\lambda} \sin(\sqrt{\lambda} x) + c_2 \sqrt{\lambda} \cos(\sqrt{\lambda} x)$$

$$\phi'(0) = c_2 \sqrt{\lambda} = 0 \Rightarrow c_2 = 0$$

$$\phi'(L) = -c_1 \sqrt{\lambda} \sin(\sqrt{\lambda} L) = 0 \nearrow c_1 = 0 \text{ (trivial case)}$$

$$\hookrightarrow \sin(\sqrt{\lambda} L) = 0$$

$$\hookrightarrow \lambda_n = \left(\frac{n\pi}{L}\right)^2, \ n \geq 1.$$

So $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $n \geq 1$, are eigenvalues,

with $\phi(x) = c_1 \cos\left(\frac{n\pi x}{L}\right)$.

• $\underline{\lambda = 0}$:

$$\left.\begin{aligned} \phi(x) &= c_1 x + c_2 \\ \phi'(0) &= c_1 = 0 \\ \phi'(L) &= c_1 = L \end{aligned}\right\}$$

So $\lambda = 0$ is an eigenvalue, with

$$\phi(x) = 1.$$

• $\underline{\lambda < 0}$: $\phi(x) = c_1 \cosh(\sqrt{\lambda} x) + c_2 \sinh(\sqrt{\lambda} x)$

$$\phi'(x) = c_1 \sqrt{\lambda} \sinh(\sqrt{\lambda} x) + c_2 \sqrt{\lambda} \cosh(\sqrt{\lambda} x)$$

$$\phi'(0) = c_2 \sqrt{\lambda} = 0 \Rightarrow c_2 = 0$$

$$\phi'(L) = c_1 \sqrt{\lambda} \sinh(\sqrt{\lambda} L) = 0 \rightarrow c_1 = 0.$$

So no negative eigenvalues.

In summary, $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $n \geq 0$, are eigenvalues, with

$$\Phi_n(x) = \cos\left(\frac{n\pi x}{L}\right) \text{ eigenfunctions.}$$

- Solution for $G(t)$:

$$G_n'(t) = -\left(\frac{n\pi}{L}\right)^2 k\, G_n(t) \rightarrow G_n(t) = c\, e^{-\left(\frac{n\pi}{L}\right)^2 k t}, \quad n \geq 0.$$

- Superposition:

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 k t}.$$

- IC:

$$u(x,0) = f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \rightsquigarrow A_n.$$

(we will see how).

→ Many other possibilities: variations of the PDE, mixed BCs, Robin BCs, ...