MATH 425 LECTURE 10

: Fourier Series I (The Coefficients)

In lecture 9 we found that the last step of the method of separation of variables needs an important assumption:

"Any function $f(x)$ can be written as an infinite expansion of sines, or cosines, or sines and cosines".

- In this lecture we will obtain the coefficients of such expansions (assuming they hold).

- Remarks:

1) The sines, cosines, or sines and cosines were the solutions to the eigenvalue problem $-\phi'' = \lambda\phi$ with Dirichlet, Neumann, and periodic BCs (respectively).

→ In lecture 11 we will study expansions with different eigenfunctions (which appear when mixing or changing the BC.).

2) In Lecture 12 we will discuss when and in which sense these expansions hold.
("Completeness")

3) While we are interested in these "Fourier series" to solve PDEs, you can see that they are a powerful tool on their own: we are "decomposing" a function into a sum of "easy" functions.
You should keep in mind the analogy with vectors: the eigenfunctions will be an (infinite) basis.

- Def.: Fourier Sine Series.

The Fourier Sine Series of $f(x)$ in $(0, L)$ is given by

$$S(f)(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right). \quad \text{with}$$

$$B_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

We will see that quite generally $f(x) = S(f)(x)$ in $(0,L)$.

Notice that the choice of the coefficients is necessary if we want the equality $f(x) = \sum_{n=1} B_n \sin\left(\frac{n\pi x}{L}\right)$ to be true.

Let's see why:

- <u>Lemma</u>: Orthogonality of Sines.

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0 & m \neq n, \\ \frac{L}{2} & m = n \neq 0. \end{cases}$$

[Proof: Trigonometric identity and compute the integral. See book page -105-]

↳ In Lect. 11 we'll see a more general prove.

Now, let's find the B_n using this orthogonality property:

$$f(x) = \sum_{n=1} B_n \sin\left(\frac{n\pi x}{L}\right) \Rightarrow \int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx = \sum_{n=1}^{\infty} B_n \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx$$

$$\Rightarrow \int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx = B_m \int_0^L \sin^2\left(\frac{m\pi x}{L}\right) dx = \frac{L}{2} B_m \Rightarrow$$

$$\Rightarrow B_m = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx \quad //.$$

• **Def:** Fourier Cosine Series

The Fourier Cosine Series of $f(x)$ in $(0,L)$ is given by

$$C(f)(x) = \sum_{n=0} A_n \cos\left(\frac{n\pi x}{L}\right), \text{ with}$$

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n \neq 0,$$

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx \quad (\text{average of } f(x) \text{ on } (0,L)).$$

As in the previous case, the choice of the A_n is necessary if we want $f(x) = C(f)(x)$.

• **Lemma:** Orthogonality of Cosines.

$$\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 0 & \text{if } n \neq m, \\ \frac{L}{2} & \text{if } n = m \neq 0, \\ L & \text{if } n = m = 0. \end{cases}$$

[similar proof. We'll see a general one.].

We repeat the reasoning to obtain A_n using this lemma:

If $f(x) = \sum\limits_{n=0} A_n \cos\left(\frac{n\pi x}{L}\right)$, then

$$\int_0^L f(x)\cos\left(\frac{m\pi x}{L}\right)dx = \int_0^L \left(\sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)\right)\cos\left(\frac{m\pi x}{L}\right)dx =$$

$$= \sum_{n=0}^{\infty} A_n \int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx \Rightarrow$$

For $m=0$: $\int_0^L f(x)dx = A_0 L \Rightarrow A_0 = \frac{1}{L}\int_0^L f(x)\,dx.$

For $m \neq 0$: $\int_0^L f(x)\cos\left(\frac{m\pi x}{L}\right)dx = A_m \int_0^L \cos^2\left(\frac{m\pi x}{L}\right)dx = A_m \frac{L}{2} \Rightarrow$

$$\Rightarrow A_m = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{m\pi x}{L}\right)dx \,//.$$

- <u>Def.</u>: (Full) Fourier Series.

The Fourier Series of $f(x)$ in $(-L,L)$ is given by

$$\mathcal{F}(f)(x) = \sum_{n=0}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right), \text{ with}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx,\ n\neq 0,\quad a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

The choice of the coefficients is justified by

- **Lemma:** Orthogonality of sines and cosines.

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = 0,$$

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0 & n \neq m, \\ L & n = m \neq 0, \\ 2L & n = m = 0, \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0 & n \neq m, \\ L & n = m \neq 0. \end{cases}$$

- **Example:** Let $f(x) = x^2$. [Even functions]

1) Find $C(f)(x)$ in $(0, L)$.

2) Find $F(f)(x)$ in $(-L, L)$.

3) Show that $\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$.

Solution:

1) $C(f)(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$,

$$A_0 = \frac{1}{L}\int_0^L x^2\,dx = \frac{1}{L}\frac{L^3}{3} = \frac{L^2}{3},$$

$$A_n = \frac{2}{L}\int_0^L x^2 \cos\left(\frac{n\pi x}{L}\right)dx = (-1)^n 4\frac{L^2}{(n\pi)^2}.$$

2) $F(f)(x) = \sum_{n=0}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$,

$$a_0 = \frac{1}{2L}\int_{-L}^{L} x^2\,dx = \frac{1}{L}\int_0^L x^2\,dx = A_0 = \frac{L^2}{3}.$$

x^2 is even on $(-L,L)$

$$a_n = \frac{1}{L}\int_{-L}^{L} x^2 \cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L x^2\cos\left(\frac{n\pi x}{L}\right)dx =$$

$$= A_n = (-1)^n 4\frac{L^2}{(n\pi)^2}.$$

$\left.\begin{array}{l} x^2 \text{ is even} \\ \cos\left(\frac{n\pi x}{L}\right) \text{ is even}\end{array}\right\} \Rightarrow x^2\cos\left(\frac{n\pi x}{L}\right)$ is even.

$$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{x^2 \sin\left(\frac{n\pi x}{L}\right)}_{\text{odd}}dx = 0.$$

Therefore,

$F(f)(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) = C(f)(x)$. Notice that this is true for all even functions.

3) Assume that in parts 1) and 2) we had that the function is equal to its Fourier expansions:

$$x^2 = \frac{L^2}{3} + \sum_{n=1}^{\infty} (-1)^n 4 \frac{L^2}{(n\pi)^2} \cos\left(\frac{n\pi x}{L}\right).$$

Clearly, the above "equality" cannot be true for all $x \in \mathbb{R}$:

$$\underbrace{x^2 - \frac{L^2}{3}}_{\text{non-periodic}} = \underbrace{\sum_{n=1}^{\infty} (-1)^n 4 \frac{L^2}{(n\pi)^2} \cos\left(\frac{n\pi x}{L}\right)}_{\text{periodic with period } 2L}$$

What will hold in general is that $x^2 = C(f)(x)$ in $(0,L)$ and $x^2 = \mathcal{F}(f)(x)$ in $(-L,L)$, and sometimes also on the boundary points (see Lecture 12).

For the moment, assume in this case that

$$x^2 = \frac{L^2}{3} + \sum_{n=1}^{\infty} (-1)^n 4 \frac{L^2}{(n\pi)^2} \cos\left(\frac{n\pi x}{L}\right) \quad \text{for } x \in [-L, L].$$

Then, when $x = L$, we obtain that

$$L^2 = \frac{L^2}{3} + \sum_{n=1}^{\infty} (-1)^n 4 \frac{L^2}{(n\pi)^2} \cos(n\pi) \Rightarrow$$

$$\Rightarrow \frac{2}{3} = \sum_{n=1}^{\infty} 4 \frac{1}{(n\pi)^2} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} //.$$

• Example: Let $f(x) = 1$ in $(0, L)$.

1) Find its Fourier Sine series.

2) Define $f_{odd}(x) = \begin{cases} 1, & 0 < x < L, \\ 0, & x = 0, \\ -1, & -L < x < 0, \end{cases}$. Find the Fourier series of f_{odd} in $(-L, L)$.

3) Find the Fourier Cosine series of $f(x)$ on $(0, L)$.

Solution:

1) $S(f)(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$ with

$$B_n = \frac{2}{L} \int_0^L 1 \sin\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L} \frac{-L}{n\pi} \cos\left(\frac{n\pi x}{L}\right)\Big]_0^L =$$

$$= \frac{2}{n\pi} (1 - (-1)^n). \text{ Thus,}$$

$$S(f)(x) = \sum_{n=1}^{\infty} 2 \frac{1-(-1)^n}{n\pi} \sin\left(\frac{n\pi x}{L}\right) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{2k-1} \sin\left(\frac{(2k-1)\pi x}{L}\right).$$

2) $\mathcal{F}(f_{odd})(x) = \sum_{n=0}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right),$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f_{odd}(x)\,dx = 0,$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f_{odd}(x)\,dx = 0,$$

$$b_n = \frac{1}{L}\underbrace{\int_{-L}^{L} f_{odd}(x)\sin\left(\frac{n\pi x}{L}\right)dx}_{\text{even}} = \frac{2}{L}\int_{0}^{L} f_{odd}(x)\sin\left(\frac{n\pi x}{L}\right)dx =$$

$$= \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = B_n.$$

That is, $\mathcal{F}(f_{odd})(x) = S(f)(x)$.

• <u>Remark</u>: It is clear that the expansions above don't converge to 1 at $x=0$ or $x=L$. Indeed,

$S(f)(0) = S(f)(L) = 0.$

Thus, we cannot expect the expansion $1 = S(f)(x)$ to hold at those points.

3) This case is trivial:

$$C(f)(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right),$$

$$A_n = \frac{2}{L}\int_0^L 1\cdot\cos\left(\frac{n\pi x}{L}\right)dx = 0 \text{ for } n \neq 0,$$

$$A_0 = \frac{1}{L}\int_0^L 1\, dx = 1.$$

That is, $C(f)(x) = 1$.

→ When a function is already a linear combination of the basis functions (eigenfunctions), then its "Fourier" expansion is the function itself.

• Summary / Conclussion:

1) The eigenvalue problems

$$\left.\begin{array}{l} -\phi'' = \lambda\phi \text{ in } (0,L) \\ \phi(0) = \phi(L) = 0 \end{array}\right\} \quad \left.\begin{array}{l} -\phi'' = \lambda\phi \text{ in } (0,L) \\ \phi'(0) = \phi'(L) = 0 \end{array}\right\} \quad \left.\begin{array}{l} -\phi'' = \lambda\phi \text{ in } (-L,L) \\ \phi(-L) = \phi(L) \\ \phi'(-L) = \phi'(L) \end{array}\right\}$$

① [Dirichlet] ② [Neumann] ③ [Periodic]

give rise to the sets of eigenfunctions

① $\lambda_n = \left(\frac{n\pi}{L}\right)^2, n \geq 1, \ \phi_n(x) = \sin\left(\frac{n\pi x}{L}\right),$

② $\lambda_n = \left(\frac{n\pi}{L}\right)^2, n \geq 0, \ \phi_n(x) = \cos\left(\frac{n\pi x}{L}\right),$

③ $\lambda_n = \left(\frac{n\pi}{L}\right)^2, n \geq 0, \ \phi_n(x) = c_1 \cos\left(\frac{n\pi x}{L}\right) + c_2 \sin\left(\frac{n\pi x}{L}\right).$

2) All these sets of eigenfunctions are orthogonal:

$$\int_{\text{domain}} \phi_n(x)\,\phi_m(x)\,dx = 0 \quad \text{for } n \neq m.$$

3) If we assume these sets are "complete", that is, that we can expand

$$f(x) = \sum_n c_n \phi_n(x),$$

the orthogonality property allows us to find c_n.

4) We gave a name to the particular series that arise in ①, ② and ③.

5) The following relationships were easy to check:

- Given $f(x)$ defined on $(0, L)$ and its even extension $f_{even}(x)$ in $(-L, L)$, we have that

$$C(f)(x) = \mathcal{F}(f_{even})(x)$$

- Given $f(x)$ in $(0, L)$ and its odd extesion $f_{odd}(x)$,

$$S(f)(x) = \mathcal{F}(f_{odd})(x)$$