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| MATH 425 |
| LECTURE 11 |

: Fourier Series II (Orthogonality)

- In this lecture we want to show that the eigenfunctions of the eigenvalue problem

$$\left.\begin{array}{l} -\phi'' = \lambda\phi \\ \text{with BCs} \end{array}\right\} \text{ are "orthogonal".}$$

This property is needed to find the coefficients of the "generalised Fourier series",

$$f(x) = \sum_n c_n \phi_n(x).$$

→ We first need to define "orthogonality" of functions.

- Def.: Inner product

The inner product of two functions in $(a,b)$ is

$$\langle f, g\rangle = \int_a^b f(x)\,\overline{g(x)}\,dx,$$

where the bar denotes complex conjugate.

• Def.: Orthogonal functions

Two functions are said to be orthogonal if

$< f, g > = 0$.

• Remark: One can check that this definition satisfies the properties of a general inner product. In particular,

$$< f, f > = \int_a^b f(x)\,\overline{f(x)}\,dx = \int_a^b |f(x)|^2\,dx \geq 0.$$

Moreover, $< f, f > = 0$ only if $f$ is the zero function.

⌈ To be precise, we should define the class of functions we are considering: for example continuous functions.

We could also consider "$L^1$ functions": ⌈ Don't worry about this. You need more analysis ⌋.

$f \in L^1(a,b)$ if $\int_a^b |f(x)|\,dx < \infty$.

When we talk about $L^1$ functions, we generally identify those functions which only differ on "isolated points", so the statement

$$\int_a^b |f(x)|^2\,dx = 0 \Rightarrow f \equiv 0$$ also makes sense. ⌋

• Remark: Once we have an inner product, we can define the norm (or "size") of a function by

$$\|f\| = \sqrt{\langle f, f \rangle} = \left( \int_a^b |f(x)|^2 \, dx \right)^{1/2}.$$

Moreover, this defines a distance between functions:

$$d(f, g) = \|f - g\|$$

All these concepts are totally analogous to the ones with column vectors.

→ When we worked with vectors in $\mathbb{R}^3$, we liked having an orthogonal basis. For example,

$$B = \{ \vec{e}_1, \vec{e}_2, \vec{e}_3 \} \text{ with } \vec{e}_i \cdot \vec{e}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases},$$

allows us to find the coefficients in the linear combination

$$\vec{u} = c_1 \vec{e}_1 + c_2 \vec{e}_2 + c_3 \vec{e}_3 \rightsquigarrow c_1 = \frac{\vec{u}_1 \cdot \vec{e}_1}{\vec{e}_1 \cdot \vec{e}_1} = \vec{u}_1 \cdot \vec{e}_1 .$$

- In linear algebra, the vector space is $\mathbb{R}^n$, and the operator are matrices. An eigenvalue problem was

$A\vec{u} = \lambda \vec{u}$.

That is, the eigenvectors are those directions in which the matrix acts as a scalar.

→ Symmetric matrices were specially nice: thanks to the Spectral Theorem, one knows that all their eigenvalues are real and the eigenvectors can be taken to be orthogonal

↳ defining an orthogonal basis of $\mathbb{R}^n$.

- Denote A the differential operator $-\frac{d^2}{dx^2}$ together with some boundary conditions on $[a,b]$.

The values of $\lambda$ for which the problem

$A\phi = \lambda\phi$ has non-zero solutions $\phi$ are called eigenvalues of A. The corresponding $\phi$ are called eigenfunctions.

(this notation includes the BCs.)

- We will focus on "symmetric" problems:

- **Def.:** Symmetric (or Hermitian) BCs

The BCs associated with $-\phi'' = \lambda\phi$ are called symmetric if

$$\left[\phi_1' \overline{\phi_2} - \phi_1 \overline{\phi_2'}\right]_a^b = 0.$$

(hermitian is used when working with complex numbers).

(symmetric is thus a particular case)

The reason for this definition will be clear in the next two propositions (we want to have the same properties than for symmetric matrices).

- **Proposition:** The eigenvalues of $-\frac{d^2}{dx^2}$ with symmetric BCs are all real. Moreover, one can always choose the eigenfunctions to be real-valued.

**Proof:** Let $\lambda \in \mathbb{C}$ be an eigenvalue, $\phi(x)$ its eigenfunction:

$$-\phi'' = \lambda\phi.$$

It is clear that $-\overline{\phi''} = \overline{\lambda}\,\overline{\phi}$, so

$$-\phi'' = \lambda\phi \rightarrow -\phi''\overline{\phi} = \lambda\phi\overline{\phi} \rightarrow -\langle \phi'', \phi \rangle = \lambda \|\phi\|^2$$

$$-\overline{\phi''} = \overline{\lambda}\,\overline{\phi} \rightarrow -\overline{\phi''}\phi = \overline{\lambda}\,\overline{\phi}\phi \rightarrow -\langle \phi, \phi'' \rangle = \overline{\lambda}\|\phi\|^2$$

Substracting both equations,

$$(\lambda-\bar{\lambda})\|\phi\|^2 = -\int_a^b \phi''(x)\overline{\phi(x)}\,dx + \int_a^b \phi(x)\overline{\phi''(x)}\,dx \, .$$

Integration by parts give that:

$$(\lambda-\bar{\lambda})\|\phi\|^2 = \int_a^b \cancel{\phi'(x)\overline{\phi'(x)}\,dx} - \phi'(x)\overline{\phi(x)}\Big]_a^b +$$

$$-\int_a^b \cancel{\phi'(x)\overline{\phi'(x)}\,dx} + \phi(x)\overline{\phi'(x)}\Big]_a^b =$$

$$= \left[\phi(x)\overline{\phi'(x)} - \overline{\phi(x)}\,\phi'(x)\right]_a^b = 0 \, .$$

$\uparrow$

thanks to the symmetric condition.

Therefore, $\lambda = \bar{\lambda}$. That is, $\lambda \in \mathbb{R}$.

[ $\|\phi\| > 0$ because $\phi$ is an eigenfunction, so $\phi \neq 0$ ].

• Now that we know that $\lambda \in \mathbb{R}$, we proceed with the second part:

$\phi(x) \in \mathbb{C} \Rightarrow \phi(x) = \operatorname{Re}\phi(x) + i \operatorname{Im}\phi(x)$.

Given that $-\phi''(x) = \lambda\phi(x)$ with $\lambda \in \mathbb{R}$, it follows that

$$-(\mathrm{Re}\,\phi(x))'' = \lambda(\mathrm{Re}\,\phi(x))$$
$$-(\mathrm{Im}\,\phi(x))'' = \lambda(\mathrm{Im}\,\phi(x))$$
so $\mathrm{Re}\,\phi(x)$ and $\mathrm{Im}\,\phi(x)$ are eigenfunctions (they also satisfy the BCs).

So instead of choosing $\phi(x)$ and $\overline{\phi(x)}$, one can take $\mathrm{Re}\,\phi(x)$ and $\mathrm{Im}\,\phi(x)$. //

→ Now we can prove the main point of this lecture:

- Proposition: Consider $-\frac{d^2}{dx^2}$ with symmetric BCs. Then, eigenfunctions corresponding to different eigenvalues are orthogonal.

  Moreover, if an eigenvalue has several (independent) eigenfunctions, one can always choose them to be

Proof: We will only prove the first part. The second part is done by the Gram-Schmidt algorithm.

↳ The interesting question would be: if an eigenvalue is "repeated", can we get "enough" eigenfunctions?

(think in eigenvectors ~ algebraic/geometric multiplicity)

Since we will deal with "completeness" in future lectures, we can forget about it now. $\Downarrow$

Consider $\lambda_1 \neq \lambda_2$, both real without loss of generality (previous Prop.), and let $\phi_1, \phi_2$ be the resp. eigenfunctions:

$$-\phi_1'' = \lambda_1 \phi_1$$

$$-\phi_2'' = \lambda_2 \phi_2 .$$

Then,

$$-\langle \phi_1'', \phi_2 \rangle = \lambda_1 \langle \phi_1, \phi_2 \rangle$$

$$-\langle \phi_2'', \phi_1 \rangle = \lambda_2 \langle \phi_2, \phi_1 \rangle .$$

Subtract both equalities,

$$(\lambda_1 - \lambda_2) \langle \phi_1, \phi_2 \rangle = -\int_a^b \phi_1''(x)\phi_2(x)\,dx + \int_a^b \phi_2''(x)\phi_1(x)\,dx =$$

$$= 0 \quad \text{(integ. by parts and symmetric B.C.)}.$$

Because $\lambda_1 \neq \lambda_2$, we deduce that

$\langle \phi_1, \phi_2 \rangle = 0$ (that is, they are orthogonal) //.

• Application: Let $\{\phi_n\}_n$ be the set of eigenfunctions from $-\phi'' = \lambda\phi$ with symmetric BC.

Then, if

$$f(x) = \sum_n c_n \phi_n(x),$$

the coefficients must be $c_n = \dfrac{\langle f, \phi_n \rangle}{\|\phi_n\|^2}$.

Check:

$$f(x) = \sum_n c_n \phi_n(x) \Rightarrow \langle f, \phi_m \rangle = \sum_n c_n \langle \phi_n, \phi_m \rangle =$$

$$= c_m \langle \phi_m, \phi_m \rangle \Rightarrow$$

$$\Rightarrow c_m = \frac{\langle f, \phi_m \rangle}{\|\phi_m\|^2}.$$

(we have used the expansion $f(x) = \sum_n c_n \phi_n(x)$ where all the eigenfunctions were chosen to be orthogonal).

• Remark: Symmetric BCs?

One can check Dirichlet, Neumann, Periodic, and Robin BCs are symmetric.

• Remark: An example where each eigenvalue has multiple eigenfunctions is the case with periodic BCs. Fortunately, we directly found orthogonal eigenfunctions (sine and cosine) without using Gram-Schmidt algorithm.

• Exercise: Show that the eigenvalues of $-\frac{d^2}{dx^2}$ with symmetric BCs, satisfying in addition $\phi(x)\phi'(x)\Big|_a^b \le 0$, cannot be negative.

Solution: Let $\lambda \in \mathbb{R}$, $\phi(x) \in \mathbb{R}$ be a pair of eigenvalue and eigenfunction (real because of symmetry).

$$-\phi''(x) = \lambda\phi(x) \Rightarrow -\langle \phi'', \phi\rangle = \lambda \|\phi\|^2 \Rightarrow$$

$$\Rightarrow \lambda\|\phi\|^2 = -\int_a^b \phi''(x)\phi(x)\,dx = \int_a^b \phi'(x)\phi'(x)\,dx - \phi'(x)\phi(x)\Big|_a^b =$$

$$= \underbrace{\|\phi'\|^2}_{\ge 0} \underbrace{- \phi'(x)\phi(x)\Big|_a^b}_{\ge 0} \ge 0 \Rightarrow \lambda \ge 0.$$

$\uparrow \|\phi\| > 0$ (eigenfunction).