$\boxed{\begin{array}{l}\text{MATH 425}\\ \text{LECTURE 12}\end{array}}$: Fourier Series III

(Statements of Convergence Results)

In lecture 11, we found that the set of eigenfunctions given by the eigenvalue problem

$$\left.\begin{array}{ll} -\phi'' = \lambda\phi & \text{in } (a,b) \\ \text{symmetric BC on } x=a,\, x=b & \end{array}\right\}$$

could always be chosen real-valued and orthogonal.

$\{\phi_n\}_n$, $\phi_n(x) \in \mathbb{R}$, $\langle \phi_n, \phi_m \rangle = 0$ if $n \neq m$.

This property determines the coefficients of the generalized Fourier Series

$$f(x) = \sum_n c_n \phi_n(x) \longrightarrow c_n = \frac{\langle f, \phi_n \rangle}{\|\phi_n\|^2}.$$

- In this lecture we focus on the question about "completeness":

"Can any function be written as a generalized Fourier Series?"

Roughly, a set of eigenfunctions is called complete if all other functions can be written as a series of them.

↳ I.e., if they form a "basis".

Of course, we need to be more precise. What do we mean by "any function"? And in which sense do we require the infinite series to be equal to the function?

For example, if we consider the orthogonal set $\{1, \sin(nx), \cos(nx)\}_{n\geq 1}$ on $[-\pi,\pi]$, (the classical full Fourier Series)

- Can all the continuous functions on $[-\pi,\pi]$ be written as a Fourier Series?
- Can all the integrable functions on $[-\pi,\pi]$ (i.e. $\int_{-\pi}^{\pi}|f(x)|dx<\infty$) be written as..? ($L^1$ functions)

- **Remark**: Notice that if we work with L^1 functions, we might not care about what happens pointwise (that is, at every point).

We may prefer a new way of considering functions to be equal. For example, we could define

$$f \underset{\text{in } (-\pi,\pi)}{=} g \quad \text{if} \quad \int_{-\pi}^{\pi}|f(x)-g(x)|dx=0.$$

• We consider here three types of convergence for infinite series of functions

(each one defines a different way of understanding the statement "$f(x) = \sum_n c_n \phi_n(x)$").

• Def.: Pointwise convergence.

The infinite series $\sum_{n=1}^{\infty} f_n(x)$ converges pointwise on (a,b) to $F(x)$ if

$$\lim_{N\to\infty} \left| F(x) - \sum_{n=1}^{N} f_n(x) \right| = 0 \quad \text{for each } x \in (a,b)$$

This seems the most natural way of convergence. We compare $F(x)$ and $\sum_{n=1}^{\infty} f_n(x)$ at each x.

Notice that the limit above is a usual limit in real numbers. That is, $\sum_{n=1}^{\infty} f_n(x)$ converges to $F(x)$ if:

$$\forall x \in (a,b),\ \forall \varepsilon > 0,\ \exists\, \underline{N = N(x,\varepsilon)} \ /$$

$$\text{for } M \geq N \rightarrow \left| F(x) - \sum_{n=1}^{M} f_n(x) \right| < \varepsilon.$$

Although this seems a good way of defining our "equal", it turns out not to be that useful. Two main problems:

→ For many functions, the (generalized) Fourier Series do not converge pointwise for all $x \in (a,b)$.

→ Having pointwise convergence does not guarantee that we can take derivatives term by term.

(etc.)

Intuition: (optional)

The rate of convergence depends on x: $N = N(x, \varepsilon)$.

So the sum might approximate very well the function $F(x)$ at $x = x_1$, while being horrible at a nearby point $x = x_2$ (so the derivative of the function would be different than that of the approximating sums).

• **Def.:** Uniform Convergence.

$\sum_{n=1}^{\infty} f_n(x)$ converges uniformly to $F(x)$ in (a,b) if

$$\lim_{N\to\infty} \sup_{x\in(a,b)} \left|F(x) - \sum_{n=1}^{N} f_n(x)\right| = 0.$$

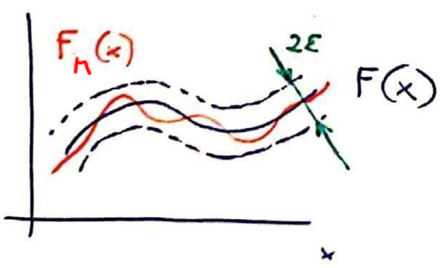
→ Notice that this definition is equivalent to:

$\forall x \in (a,b),\ \forall \varepsilon > 0,\ \exists\ \underline{N = N(\varepsilon)}$ /

← independent of x.

for $M \geq N \rightarrow \left|F(x) - \sum_{n=1}^{M} f_n(x)\right| < \varepsilon$.

We can see that for each M, the approximating series $F_M(x) = \sum_{n=1}^{M} f_n(x)$ is within a band of width ε with respect to $F(x)$:



This is a much stronger convergence than pointwise! Indeed, if a series converges uniformly in (a,b), then it also converges pointwise in (a,b) (the converse is false).

• Example: Consider $f_n(x) = (1-x)x^{n-1}$ on $(0,1)$.

• The partial sums are $F_N(x) = \sum_{n=1}^{N} f_n(x) =$

$$= \sum_{n=1}^{N} (x^{n-1} - x^n) = 1 - x^N.$$

Therefore, the pointwise limit is $\lim_{N\to\infty} F_N(x) = 1$ for $x \in (0,1)$.

That is, $\sum_{n=1}^{\infty} f_n(x)$ converges to $F(x) = 1$ pointwise in $(0,1)$.

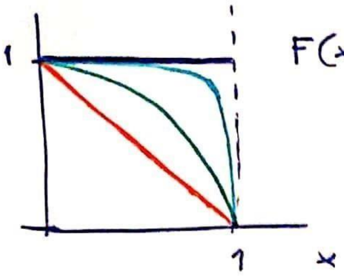
• However,

$$\sup_{x\in(0,1)} |F(x) - F_N(x)| = \sup_{x\in(0,1)} |1 - (1-x^N)| = \sup_{x\in(0,1)} |x^N| = 1.$$

Thus, $\sum_{n=1}^{\infty} f_n(x)$ does not converge uniformly to 1 in $(0,1)$.

What is happening? Notice that in the uniform convergence the boundary points matter because in the definition there is a supremum. Let's see a picture:

$F_N(x) = 1 - x^N$



Near $x = 1$, it is impossible to find a band of width ε where $F_N(x)$ fits.

$\downarrow$

However, for all $x \in (0,1)$, taking N sufficiently big, $F_N(x)$ gets close enough to 1.

- Sometimes (most times) is difficult to check if a series converges uniformly using the definition. A useful test is the following:

- Weiertrass Test:

If $|f_n(x)| \leq c_n$ for all n and $x \in (a,b)$, and if $\sum_{n=1}^{\infty} c_n$ converges, then $\sum_{n=1}^{\infty} f_n(x)$ converges unif. in (a,b).

→ Remember the convergence tests for numerical series. We will only use one:

Comparison with $\sum_{n=1}^{\infty} \frac{1}{n^s}$ ↘

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{converges if } s>1, \text{ diverges if } s\leq 1.$$

So if our functions $f_n(x)$ decay faster than $\frac{1}{n^{1+\varepsilon}}$ ($\varepsilon>0$), that is,

$|f_n(x)| \leq \frac{1}{n^{1+\varepsilon}}$, then $\sum_{n=1}^{\infty} f_n(x)$ converges unif. $\lrcorner$

- Uniform convergence is very nice (we can differentiate term by term; if the $f_n(x)$ are continuous, the the limit function is continuous, ...) but it is too strong: many functions don't have unif. convergent Fourier Series.

- Remark: If $F(x)$ is discontinuous and the functions $f_n(x)$ are continuous, the $\sum_{n=1}^{\infty} f_n(x)$ cannot converge unif. to $F(x)$.

[The proof consists in choosing the appropriate ε-δ. We won't show it, but it is not hard if you are comfortable with analysis].

We finish with a weaker definition of convergence, fundamental for Fourier Series : L^2 convergence.
(this is also very important in physics, as it is the natural sets of functions to work with in most situations : finite energy fluid mechanics, quantum mechanics, etc.)

• <u>Def.</u>: L^2 convergence

$\sum_{n=1}^{\infty} f_n(x)$ converges to $F(x)$ in $L^2(a,b)$ if

$$\lim_{N\to\infty} \int_a^b \left|F(x) - \sum_{n=1}^{N} f_n(x)\right|^2 dx = 0 .$$

→ In Lect. 11 we define an inner product of functions:

$$\langle f, g\rangle = \int_a^b f(x)\,\overline{g(x)}\,dx,$$

which gave rise to a norm $\|f\| = \sqrt{\langle f, f\rangle}$, and correspondingly to a distance $d(f,g) = \|f-g\|$.

This is the so-called L^2 norm, that we will denote from now on by $\|\cdot\|_{L^2}$:

$$\|f\|_{L^2} = \left(\int_a^b |f(x)|^2 \, dx\right)^{1/2}.$$

So the L^2 convergence in (a,b) says:

$\sum_{n=1}^{\infty} f_n(x)$ converges to $F(x)$ in $L^2(a,b)$ if the distance between $F(x)$ and the approximating function $F_N(x) = \sum_{n=1}^{N} f_n(x)$ goes to zero as $N \to \infty$:

$$\lim_{N\to\infty} \| F - F_N \|_{L^2} = 0.$$

This language applies to the other modes of convergence:

1) Pointwise convergence: typical convergence of real numbers (for each x)

2) Uniform convergence: it can be checked that $\sup_{x \in (a,b)} |f(x)|$ defines a norm, called the L^∞ norm.

$$\|f\|_{L^\infty} = \sup_{x\in(a,b)} |f(x)|.$$

• **Remark:** 1) Unif. convergence $\overset{\not\Leftarrow}{\Rightarrow}$ Pointwise convergence.

2) Unif. convergence in (a,b) $\overset{\not\Leftarrow}{\Rightarrow}$ L^2 convergence in (a,b) (if (a,b) is bounded)

3) Pointwise $\underset{\not\Leftarrow}{\not\Rightarrow}$ L^2 convergence.

• Finally, let's state some theorems about the convergence of the Fourier Series.

↳ There are many more (with more general conditions).
↳ The proofs can be highly non-trivial.
  ↳ Specially for pointwise convergence.

• **Theorem 1:** Uniform Convergence

The generalized Fourier Series $\sum_n c_n \phi_n(x)$ converges uniformly on (a,b) to $f(x)$ if

1) $f(x), f'(x), f''(x)$ exist and are continuous on $[a,b]$.

2) $f(x)$ satisfy the boundary conditions (that give rise to $\phi_n(x)$).

• Remarks:

1) All the theorems will refer to ϕ_n coming from $-\phi'' = \lambda\phi$ with symmetric BCs. The coeff. are $c_n = \dfrac{\langle f, \phi_n \rangle}{\|\phi_n\|^2}$.

2) For the classical full Fourier Series, one only needs $f(x)$ continuous on $[a,b]$ and $f'(x)$ piecewise continuous.
↳ That is, sines and cosines.

[This is one reason why this expansion is usually preferred].

3) For the classical Fourier Series (full, sine, cosine), $f''(x)$ doesn't need to exist.

4) If the theorem fails to apply, one can try with Weierstrass test.

- **Theorem 2**: Completeness in L^2

If $\|f\|_{L^2} < +\infty$, then the series $\sum_n c_n \phi_n$ converges in (a,b) to $f(x)$ in L^2 sense.

That is,

$\|$ Any function in L^2 has a convergent Fourier Series in L^2 $\|$

- **Theorem 3**: Pointwise convergence of Classical Fourier Series.

Let $f(x)$, $f'(x)$ be piecewise continuous functions on $[a,b]$.

Then, the (full, sine, cosine) Fourier series converges to

$$\sum_n c_n \phi_n(x) = \frac{1}{2}\left(f(x^+) + f(x^-)\right) \quad \text{for all } x \in (a,b),$$

where $f(x^+)$ is the limit from the right side and $f(x^-)$ the limit from the left.

- **Remark**: Considering the corresponding extension to the whole line, then the sum converges to $\frac{1}{2}\left(f_{ext}(x^+) + f_{ext}(x^-)\right)$ for all $x \in \mathbb{R}$.

- Example: $f(x) = x^2$, $x \in (0,L)$. Its Fourier Cosine Series is

$$C(f)(x) = \frac{L^2}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{L^2}{n^2\pi^2} \cos\left(\frac{n\pi x}{L}\right).$$

1) Pointwise convergence on $(0,L)$?

2) Does the series converges at $x=0$, $x=L$ to $f(0)$, $f(L)$?

3) Uniform convergence?

Solution:

1) $f(x) = x^2$, $f'(x) = 2x$ continuous on $[0,L] \Rightarrow$

$\Rightarrow$ pointwise convergent on $(0,L)$. That is, we can guarantee that

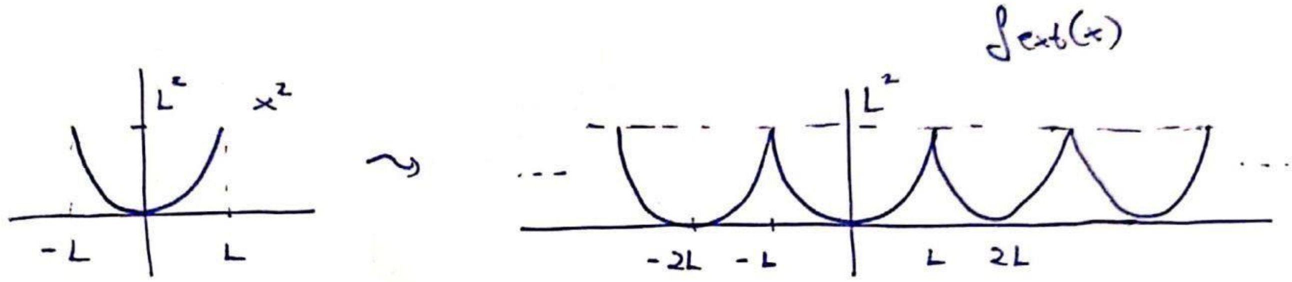
$x^2 = C(f)(x)$ for all $x \in (0,L)$.

2) Because $f(x) = x^2$ is even, we know that the full Fourier series of $f(x) = x^2$ on $(-L,L)$ is $\mathcal{F}(f)(x) = C(f)(x)$, so

$$x^2 = \frac{L^2}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{L^2}{n^2\pi^2} \cos\left(\frac{n\pi x}{L}\right) \quad \text{for } x \in (-L,L).$$

In particular, we have deduced the pointwise convergence for $x=0$.

Moreover, we can consider the periodic extension of $f(x)=x^2$ to all $\mathbb{R}$:



Theorem 3 guarantees that

$$\mathcal{F}(f)(x) = \frac{L^2}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{L^2}{n^2\pi^2} \cos\left(\frac{n\pi x}{L}\right) = \frac{1}{2}\left(f_{ext}(x^+) + f_{ext}(x^-)\right)$$

for all $x \in \mathbb{R}$.

In particular, since $f_{ext}(x)$ is continuous,

$$\mathcal{F}(f)(x) = f_{ext}(x).$$

Therefore, $\mathcal{F}(f)(L) = f_{ext}(L) = f(L)$ //.

3) $f(x)$ continuous on $[-L,L]$
$f'(x)$ piecewise continuous $\Big| \to$ uniform convergence on $(-L,L)$

$\uparrow$ Remark 2) of Theorem 1.

Also: Weierstrass test

$$\left.\begin{aligned} f_0(x) &= \frac{L^2}{3} \\ f_n(x) &= (-1)^n \frac{L^2}{n^2\pi^2}\cos\left(\frac{n\pi x}{L}\right) \end{aligned}\right\} \to |f_n(x)| \leq c_n = \begin{cases} \frac{L^2}{3}, & n=0 \\ \frac{L^2}{n^2\pi^2}, & n \geq 1 \end{cases}$$

Clearly,

$$\sum_{n=1}^{\infty} c_n = \frac{L^2}{3} + \sum_{n=1}^{\infty} \frac{L^2}{n^2\pi^2} \text{ converges} \Rightarrow F(f)(x) = \sum_{n=0}^{\infty} f_n(x)$$

converges unif.

[Indeed, this proves uniform convergence of $F(f)(x)$ to $f_{ext}(x)$ on $\mathbb{R}$].

• <u>Example:</u> $f(x) = 1$ on $(0,L)$. Fourier Sine series:

$$S(f)(x) = \sum_{k=1}^{\infty} \frac{4}{\pi} \frac{1}{2k-1} \sin\left(\frac{(2k-1)\pi x}{L}\right).$$

1) What is the value of $S(f)(x)$ for each $x \in \mathbb{R}$?

2) Is it uniformly convergent on $(0,L)$?

3) Is it convergent to L^2 on $(0,L)$?

<u>Solution:</u>

1) $\left.\begin{array}{l} f(x) = 1 \\ f'(x) = 0 \end{array}\right\} \rightarrow$ $S(f)(x)$ converges pointwise to $f(x)$ in $(0,L)$.

Now, for $x = 0$ or $x = L$, $S(f)(x) = 0$.

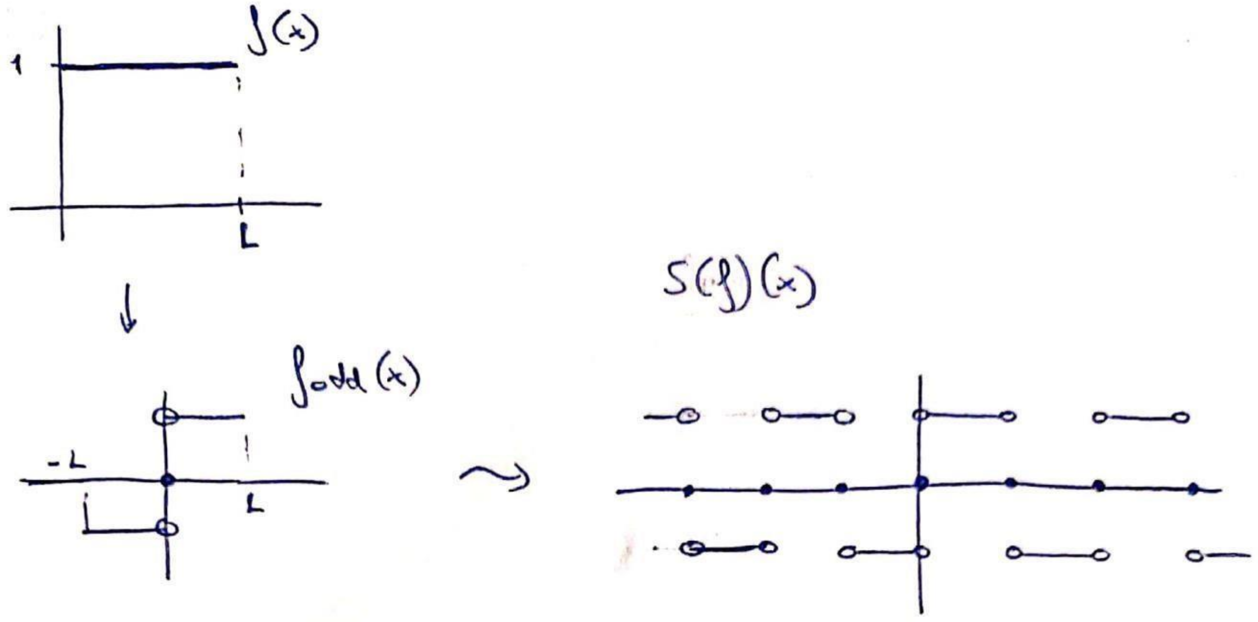
That is,

$$S(f)(x) = \begin{cases} 1 & x \in (0,L) \\ 0 & x = 0,\ x = L. \end{cases}$$

And outside $[0,L]$?

We know that $S(f)(x) = \mathcal{F}(f_{odd})(x)$.

Since f_{odd}, f'_{odd} are piecewise continuous on $[0,L]$, theorem 3 applies:



2) • Theorem 2) does not apply ($f(x)$ doesn't satisfy the BCs : $f(0) \neq 0$, $f(L) \neq 0$).

• Weierstrass test doesn't apply ($\sum \frac{1}{n}$ diverges, so no conclussion).

However, the limit is discontinuous (part 1) while each of the terms $\sin\left(\frac{(2k-1)\pi x}{L}\right)$ is continuous. Therefore, the series cannot be uniformly convergent.

3) Yes, because $f(x) = 1$ is in $L^2(0,L)$: $\int_0^L (f(x))^2 dx = L^2 < +\infty$.