MATH 241 LECTURE 12	: Sturm-Liouville Theory II.

Last day: Regular Sturm-Liouville Eigenvalue Problem (standard form)

ODE: $\frac{d}{dx}\left(p(x)\,\phi'(x)\right) + q(x)\phi(x) + \lambda\sigma(x)\phi(x) = 0, \; x \in (a,b),$

BC: $a_1\,\phi(a) + b_1\,\phi'(a) = 0$ $\quad (a_1, b_1, a_2, b_2 \in \mathbb{R})$.

$a_2\,\phi(b) + b_2\,\phi'(b) = 0$

$+$ p, q, σ real-valued and continuous functions on $[a,b]$.

$\left.\begin{array}{l} p(x) > 0 \\ \sigma(x) > 0 \end{array}\right\}$ for all $x \in [a,b]$ $\longleftarrow$ The end points are included.

• Exercise: Decide if the following problems are S-L problems in standard form. If they are not, can they be transformed into S-L in standard form?

1) $\phi''(x) + (\lambda - x^2)\phi(x) = 0$
$\phi'(0) = \phi'(1) = 0$

We easily see that this is a regular S-L problem with

$p(x) = 1$, $q(x) = -x^2$, $\sigma(x) = 1$, on $x \in [0,1]$.

- Notice that the terms with derivatives determine $p(x)$, the one with λ gives $\sigma(x)$, and the one without λ and derivatives gives $q(x)$.

[The BC are homogeneous Neumann ✓].

2) $\phi''(x) - 2x\phi'(x) + \lambda(x^2+1)\phi(x) = 0$
$\phi(0) = 0$, $\phi'(2) = 0$

Notice that for a regular S-L problem in standard form, the term with derivatives is

$$\frac{d}{dx}(p(x)\phi'(x)) = p'(x)\phi'(x) + p(x)\phi''(x).$$

Here, we can check that our ODE is not in this form:

$\phi''(x) - 2x\phi'(x) \rightsquigarrow \begin{matrix} p(x) = 1 \\ p'(x) = -2x \end{matrix}$!! Not possible.

We would like to write $\phi''(x) - 2x\phi'(x)$ as a derivative, like $\frac{d}{dx}(p(x)\phi'(x))$, but we've just seen we can't.

→ However, we can transform the given ODE into another ODE which is S-L in standard form.

- Multiply the ODE by an integrating factor $H(x)$:

$$H(x)\phi''(x) - 2x\,H(x)\phi'(x) + \lambda H(x)(x^2+1)\phi(x) = 0 \quad [*]$$

and compare to

$$p(x)\phi''(x) + p'(x)\phi'(x) + q(x)\phi(x) + \lambda\sigma(x)\phi(x) = 0.$$

If we think of $H(x)$ as $p(x)$, we would like that

$H'(x) = -2x\,H(x)$.

So, if we choose $H(x)$ as a function that solves this ODE $H'(x) = -2x\,H(x)$, then the equation [*] is a S-L ODE in standard form with

$p(x) = H(x)$,

$q(x) = 0$

$\sigma(x) = H(x)(x^2+1)$.

• Remark: For the problem to be a regular S-L problem, we need to make sure that $p(x) > 0$, $\sigma(x) > 0$ for $x \in [0, 2]$.

[The BC are ok ✓].

Let's find $H(x)$:

$$H'(x) = -2x\, H(x) \Rightarrow \frac{H'(x)}{H(x)} = -2x \Rightarrow \int \frac{dH}{H} = \int -2x\, dx \Rightarrow$$

$$\Rightarrow \log|H(x)| = -x^2 + c \Rightarrow |H(x)| = c\, e^{-x^2}.$$

We can choose a particular $H(x)$; since we want $p(x) = H(x) > 0$, let's choose $H(x) = e^{-x^2}$.

3) $$\left.\begin{aligned} & x^2 \phi''(x) + x\phi'(x) + \lambda\phi(x) = 0 \\ & \phi(1) = \phi(e) = 0 \end{aligned}\right\}$$

The ODE is not in standard S-L form. We try to transform it using an integrating factor.

$$\underbrace{H(x)\, x^2 \phi''(x) + x\, H(x)\phi'(x)} + \lambda H(x)\phi(x) = 0$$

$$= \frac{d}{dx}\left(x^2 H(x)\phi'(x)\right) \quad \text{if} \quad \frac{d}{dx}\left(x^2 H(x)\right) = x\, H(x).$$

That is, if $2x H(x) + x^2 H'(x) = x H(x)$, then our ODE

$$H(x) x^2 \phi''(x) + x H(x) \phi'(x) + \lambda H(x) \phi(x) = 0$$

is in standard S-L form with

$$p(x) = x^2 H(x), \quad q(x) = 0, \quad \sigma(x) = H(x).$$

Let's thus solve the ODE for $H(x)$:

$$2x H(x) + x^2 H'(x) = x H(x) \Rightarrow H'(x) = \frac{-1}{x} H(x) \Rightarrow$$

$$\Rightarrow \int \frac{dH}{H} = - \int \frac{dx}{x} \Rightarrow \log |H(x)| = - \log |x| + c \Rightarrow$$

$$\Rightarrow |H(x)| = \frac{c}{|x|} \rightarrow \text{We choose } H(x) = \frac{1}{x}.$$

Then, $p(x) = x, \ q(x) = 0, \ \sigma(x) = \frac{1}{x}$, with $x \in [1, e]$.

The BC. are Dirichlet and homogeneous. Therefore, our problem is now a regular S-L problem in standard form.

- <u>Remark</u>: The condition $p(x) > 0, \sigma(x) > 0$ must hold in the domain of the given problem (here, $x \in [1, e]$).

• The important point to remember is: Many real (linear, homogeneous) PDEs give rise to an eigenvalue problem when we use the method of separation of variables. In most cases, these eigenvalues problems cannot be solved explicitly (like we used to do with $\phi'' + \lambda\phi = 0$). However, for regular S-L eigenvalue problems, most of the nice properties we had still hold.

In particular, for any regular S-L problem:

1] All λ are real.

2] There exists a smallest eigenvalue λ_1, and an infinite number of eigenvalues $\lambda_1 < \lambda_2 < \dots < \lambda_n < \dots$. Moreover, there is not an upper eigenvalue, i.e. $\lambda_n \to +\infty$ as $n \to +\infty$.

3] Each eigenvalue λ_n has a unique (up to a multiplicative constant) eigenfunction ϕ_n, and this ϕ_n has <u>exactly</u> $n-1$ zeros in (a,b). (Open interval!)

↳ So for example the eigenfunction corresponding to the smallest eigenvalue has no zeros in (a,b).

Remark: Up to a multiplicative constant means that if ϕ_n is an eigenfunction, $c\,\phi_n$ is also an eigenfunction with $c \in \mathbb{R}$.

[Notice that this is always true, because the eigenfunctions are solutions to an homogeneous problem].

4] The eigenfunctions form a complete set: any piecewise smooth function can be represented as a generalized Fourier series:
$$f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x).$$

Remark: This point was crucial (when solving a PDE by separation of variables) to be able to consider any initial data.

5] Eigenfunctions corresponding to different eigenvalues are orthogonal with weight $\sigma(x)$, that is,
$$\int_a^b \phi_n(x)\phi_m(x)\,\sigma(x)\,dx = 0 \quad \text{for } n \neq m.$$

→ Crucial to find the coefficients of the gen. Fourier Series.

5] Each λ can be defined through its eigenfunction ϕ_n:

$$\lambda = \frac{-p(x)\phi(x)\phi'(x)\Big]_a^b + \int_a^b \left(p(x)(\phi'(x))^2 - q(x)(\phi(x))^2\right)dx}{\int_a^b (\phi(x))^2 \sigma(x)\,dx}$$

↳ Rayleigh Coefficient.

<u>Remarks:</u>

- Since we will not know (in general) the eigenfunctions, we cannot really use the Rayleigh Coeff. to find the λ.
- However, we will see (through some examples) how to deduce some properties about the eigenvalues. Typically, the Rayleigh Coeff. can be used to show that $\lambda \geq \dots$ or $\lambda > \dots$.
 (remember property 1] ⇝ if we prove $\lambda_1 > \dots$, then that inequality holds for all the other eigenvalues).
- In practice ("real life"), these theorems combined with numerical methods that allow to actually (not for this course)

compute the eigenvalues using the Rayleigh Coeff.
Indeed, the procedure is:

- Without knowing the ϕ_n, use the Rayleigh Coeff. to compute the first eigenvalues (usually one just need a few of them).
- With these values of λ_n $(n=1,2,\dots,k)$, solve (numerically) the ODE (eigenvalue) problem to compute ϕ_n.
- Use these first k eigenfunctions to approximate your solution.

[see Section 5.6 and 5.10 if you are interested].

how to compute the λ by Rayleigh Coeff. using optimization

"finite Fourier Series" (approximation with only N eigenfunctions, error, and mean-square deviation).

- Let's see how to use all these properties to solve a problem.

<u>Example</u>: Wave Equation with non-constant coefficients.

PDE: $\rho(x)\, u_{tt}(x,t) = u_{xx}(x,t) + \alpha(x) u(x,t), \quad 0<x<L,\ t>0,$

BCs: $u(0,t) = u(L,t)$ ← Dirichlet

ICs: $u(x,0) = f(x)$

$u_t(x,0) = g(x)$.

(you can practice with this problem and different BC.).

Find $u(x,t)$ in terms of the eigenfunctions of the problem. Assume that $\rho(x)>0$ and $\alpha(x)>0$.

<u>Solution</u>:

We start as usual: $u(x,t) = \phi(x) G(t) \rightsquigarrow$

$$\rho(x)\phi(x)G''(t) = \phi''(x)G(t) + \alpha(x)\phi(x)G(t) \rightsquigarrow$$

$$\frac{G''(t)}{G(t)} = \frac{\phi''(x)}{\rho(x)\phi(x)} + \frac{\alpha(x)}{\rho(x)} = -\lambda.$$

<u>BC</u>: $u(0,t) = \phi(0)G(t) = 0$ for all $t\geq 0 \rightsquigarrow \phi(0)=0,$

$u(L,t) = \phi(L)G(t) = 0$ for all $t\geq 0 \rightsquigarrow \phi(L) = 0.$

① $\phi''(x) + \alpha(x)\phi(x) = -\lambda \rho(x)\phi(x), \quad 0 < x < L$

$\phi(0) = \phi(L) = 0$

② $G''(t) = -\lambda G(t)$

- We first solve the eigenvalue problem. It is a regular S-L eigenvalue problem (with $p(x)=1$, $q(x)=\alpha(x)$, $\sigma(x)=\rho(x)$), so

$\phi_n(x) \leftrightarrow \lambda_n, \quad n \geq 1.$

- With these λ_n, we solve ②. But the solutions to ② depend on whether λ_n is positive, zero or negative!

Notice that the PDE is a (modified) wave equation, for which we don't expect exponential growth in time. So we expect $\lambda \geq 0$. Moreover, because $\alpha(x) < 0$, the term $\alpha(x)u(x,t)$ is a restoring force, so we expect the solution to decay in time towards the equilibrium solution ($u \equiv 0$). That is, we expect $\lambda > 0$

We will use the Rayleigh coefficient to show that all the eigenvalues are strictly positive:

$$\lambda = \frac{-p\phi\phi'\Big]_0^L + \int_0^L (p(\phi')^2 - q\phi^2)dx}{\int_0^L \phi^2 \sigma \, dx} = \begin{cases} p(x) = 1, \\ q(x) = \alpha(x), \\ \sigma(x) = \rho(x) \end{cases}$$

$$= \frac{\int_0^L (\phi'(x))^2 dx - \int_0^L \alpha(x)(\phi(x))^2 dx}{\int_0^L (\phi(x))^2 \rho(x) \, dx}$$, where we have used the BCs.

Notice that the denominator is strictly positive since $\rho(x) > 0$ (by assumption) and $\phi \not\equiv 0$ because it is an eigenfunction.

The numerator is clearly non-negative, since $\alpha(x) < 0$.

Thus, $\lambda \geq 0$.

→ We can also prove that $\lambda \neq 0$ (that is, that $\lambda > 0$). We proceed by contradiction. Assume that $\lambda = 0$. Then,

$$\underbrace{\int_0^L (\phi'(x))^2 dx}_{\geq 0} - \underbrace{\int_0^L \alpha(x)(\phi(x))^2 dx}_{\geq 0} = 0 \Longrightarrow$$

$$\int_0^L (\phi'(x))^2\,dx = 0 \quad \text{and} \quad \int_0^L \alpha(x)(\phi(x))^2\,dx = 0.$$

$\ulcorner$ Think of this: $|x| + |y| = 0 \Rightarrow \begin{array}{l} |x| = 0 \\ |y| = 0 \end{array}$ $\lrcorner$

Then, we deduce that

$$\int_0^L (\phi'(x))^2\,dx = 0 \Leftrightarrow (\phi'(x))^2 = 0 \text{ for all } x \in [0, L].$$

That is, $\phi'(x) = 0 \Rightarrow \phi(x) = c$.

But we know that $\phi(0) = 0$, so $c = 0 \Rightarrow \phi \equiv 0$.

This is a contradiction because ϕ is an eigenfunction, so it cannot be identically zero.

Therefore, our assumption $\lambda = 0$ is wrong. In summary, $\lambda \neq 0$.

<u>Remark</u>: 1) We are assuming that $\phi(x)$ is continuous.

2) We could have used the other equation,

$$\int_0^L \underbrace{\alpha(x)}_{<0}(\phi(x))^2 = 0 \Rightarrow (\phi(x))^2 = 0 \text{ for all } x \in [0, L] \ldots$$

(← fixed sign)

Remark: $\int_a^b h(x)\,dx = 0 \not\Rightarrow h(x) = 0$.

$\int_a^b h(x)\,dx = 0 \longrightarrow h(x) = 0$.

with $h(x)$ of fixed sign (i.e. $h(x) \geq 0$ in $[a,b]$, or $h(x) \leq 0$ in $[a,b]$)

(• Think of the integral as the area below the graph.)

• Now that we know that $\lambda > 0$, we can go back to solve ②.

$G_n''(t) = -\lambda_n G_n(t)$

[I will use the subindex n to denote we are solving the problem with given λ_n]

$\downarrow$

$G_n(t) = A_n \cos(\sqrt{\lambda_n}\, t) + B_n \sin(\sqrt{\lambda_n}\, t)$.

• Product solutions: $u_n(x,t) = \phi_n(x) G_n(t)$.

• Superposition: $u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \phi_n(x) G_n(t) =$

$$= \sum_{n=1}^{\infty} \left(A_n \cos(\sqrt{\lambda_n}\, t) + B_n \sin(\sqrt{\lambda_n}\, t)\right) \phi_n(x)$$

• Initial conditions:

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \Phi_n(x),$$

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} B_n \sqrt{\lambda_n}\, \Phi_n(x).$$

Finally, we use the orthogonality property to compute the constants A_n, B_n:

$$\int_0^L f(x)\Phi_m(x)\rho(x)\,dx = \sum_{n=1}^{\infty} A_n \int_0^L \Phi_n(x)\Phi_m(x)\rho(x)\,dx =$$

$$= A_m \int_0^L (\Phi_m(x))^2 \rho(x)\,dx \Rightarrow$$

$$\Rightarrow A_m = \frac{\int_0^L f(x)\Phi_m(x)\rho(x)\,dx}{\int_0^L (\Phi_m(x))^2\rho(x)\,dx} \,//.$$

$$\int_0^L g(x)\Phi_m(x)\rho(x)\,dx = \sum_{n=1}^{\infty} B_n\sqrt{\lambda_n}\int_0^L \Phi_n(x)\Phi_m(x)\rho(x)\,dx =$$

$$= B_m\sqrt{\lambda_m}\int_0^L (\Phi_m(x))^2\rho(x)\,dx \Rightarrow$$

$$\Rightarrow B_m = \frac{\int_0^L g(x)\Phi_m(x)\rho(x)\,dx}{\sqrt{\lambda_m}\int_0^L (\Phi_m(x))^2\rho(x)\,dx} \,//.$$