MATH 425 LECTURE 13 : Fourier Series IV (Completeness, L^2 Theory)

• Context: Consider the set of eigenfunctions $\{\phi_n\}_{n\geq 1}$ given by

$$\left.\begin{array}{l} -\phi'' = \lambda\phi \quad \text{in } (a,b) \\ \text{with symmetric BCs} \end{array}\right\}$$

We were discussing when and in which sense we can write

$$f(x) = \sum_{n\geq 1} c_n \phi_n(x).$$

We already know that

1) The coefficients must be $c_n = \dfrac{\langle f, \phi_n \rangle}{\|\phi_n\|_{L^2}^2} = \dfrac{\int_a^b f(x)\phi_n(x)\,dx}{\int_a^b (\phi_n(x))^2\,dx}$.

2) Some theorems for pointwise, uniform and L^2 convergence of $\sum_{n\geq 1} c_n \phi_n(x)$ to $f(x)$.

[We will consider from now on that the coefficient are the ones for generalized Fourier Series $c_n = \dfrac{\langle f, \phi_n\rangle}{\|\phi_n\|_{L^2}^2}$, unless stated otherwise.]

→ In this lecture we focus on the L^2 convergence.

- We want to prove the Completeness theorem in L^2:

"If $f \in L^2$, then $\sum_{n \geq 1} c_n \phi_n(x)$ converges to f in L^2".

To simplify the language, we will say that two functions in L^2 are equal if their distance in L^2 is zero:

$$f = g \quad \text{in } L^2 \text{ sense} \quad \text{if} \quad \|f-g\|_{L^2} = \left(\int_a^b |f(x) - g(x)|^2 \, dx \right)^{1/2} = 0.$$

So the previous theorem can be written as

- <u>Theorem</u>: $\underbrace{f \in L^2}_{(\|f\|_{L^2} < +\infty)} \Rightarrow f = \sum_{n=1}^{\infty} c_n \phi_n$ (in L^2 sense).

$$\left[c_n = \frac{\langle f, \phi_n \rangle}{\|\phi_n\|_{L^2}^2} \right].$$

- To prove this fundamental result, we first show a theorem about the approximation of $f(x)$ by a "finite Fourier Series". This result is very important by itself (in particular in numerics).

- **Theorem:** Least-Squares Approximation.

Let $\{\varphi_n\}_{n\geq 1}$ be <u>any</u> orthogonal set of functions in L^2.
Let $f \in L^2$ and N be a positive integer.
Then,
the solution to $\displaystyle\min_{a_1,\dots,a_N} \left\| f - \sum_{n=1}^{N} a_n \varphi_n \right\|_{L^2}$ is given

by

$$a_n = \frac{\langle f, \varphi_n \rangle}{\|\varphi_n\|_{L^2}^2}.$$

- **Remarks:**

1) We will prove the theorem when φ_n are real-valued and $a_i \in \mathbb{R}$ (it is very similar in the general complex case).

↳ Notice that φ_n is any orthogonal set. It doesn't need to come from an eigenvalue problem.

2) In plain words, this theorem says: "Given a finite set of orthogonal functions $\{\varphi_n\}$, the linear combination that best approximate $f \in L^2$ in L^2 sense is the "finite Fourier series"".

• **Proof:** The idea of the proof is straightforward:

Compute the ("least-square") error $\|f - \sum_{n=1}^{N} a_n \varphi_n\|_{L^2}^2$ and we'll find the best a_n.

(for simplicity in computations)

$$E_N = \|f - \sum_{n=1}^{N} a_n \varphi_n\|_{L^2}^2 =$$

changed index to clarify later the product of the series.

$$= \langle f - \sum_{n=1}^{N} a_n \varphi_n , f - \sum_{m=1}^{N} a_m \varphi_m \rangle =$$

$$= \langle f, f \rangle - \langle \sum_{n=1}^{N} a_n \varphi_n , f \rangle - \langle f, \sum_{m=1}^{N} a_m \varphi_m \rangle +$$

$$+ \langle \sum_{n=1}^{N} a_n \varphi_n , \sum_{m=1}^{N} a_m \varphi_m \rangle =$$

(slightly different if $\varphi_n(x) \in \mathbb{C}$, $a_n \in \mathbb{C}$)

$$= \|f\|_{L^2}^2 - 2\int_a^b f(x) \Big(\sum_{n=1}^{N} a_n \varphi_n(x)\Big) dx + \int_a^b \Big(\sum_{n=1}^{N} a_n \varphi_n(x)\Big)\Big(\sum_{m=1}^{N} a_m \varphi_m(x)\Big) dx =$$

$$= \|f\|_{L^2}^2 - 2\sum_{n=1}^{N} a_n \langle f, \varphi_n \rangle + \int_a^b \sum_{n=1}^{N} \sum_{m=1}^{N} a_n a_m \varphi_n(x) \varphi_m(x)\, dx =$$

$$= \|f\|_{L^2}^2 - 2\sum_{n=1}^{N} a_n \langle f, \varphi_n \rangle + \sum_{n=1}^{N} \sum_{m=1}^{N} a_n a_m \overbrace{\langle \varphi_n , \varphi_m \rangle}^{\neq 0 \text{ only if } m=n} =$$

$$= \|f\|_{L^2}^2 - 2\sum_{n=1}^{N} a_n \langle f, \varphi_n \rangle + \sum_{n=1}^{N} a_n^2 \|\varphi_n\|_{L^2}^2 .$$

Let's write the last expression completing the square for a_n:

$$E_N = \|f\|_{L^2}^2 + \sum_{n=1}^{N} \|\varphi_n\|_{L^2}^2 \left(a_n^2 - 2a_n \frac{\langle f, \varphi_n \rangle}{\|\varphi_n\|_{L^2}^2} \right) =$$

$$= \underbrace{\|f\|_{L^2}^2}_{} + \underbrace{\sum_{n=1}^{N} \|\varphi_n\|_{L^2}^2 \left(a_n - \frac{\langle f, \varphi_n \rangle}{\|\varphi_n\|_{L^2}^2} \right)^2}_{\geq 0} - \underbrace{\sum_{n=1}^{N} \frac{(\langle f, \varphi_n \rangle)^2}{\|\varphi_n\|_{L^2}^2}}_{}.$$

[independent of a_n]

So if we want to minimize the "squared error" E_N, we must choose

$$a_n = \frac{\langle f, \varphi_n \rangle}{\|\varphi_n\|_{L^2}^2} \quad \text{for all } n \in \{1, 2, \dots, N\}. \quad //$$

• Remark: Along the proof, we have discovered an important inequality:

$$E_N = \|f\|_{L^2}^2 - \sum_{n=1}^{N} \left(\frac{\langle f, \varphi_n \rangle}{\|\varphi_n\|_{L^2}} \right)^2 \geq 0 \longrightarrow$$

with $a_n = \dfrac{\langle f, \varphi_n \rangle}{\|\varphi_n\|_{L^2}^2}$

Remember that $E_N = \| f - \sum_{n=1}^{N} a_n \varphi_n \|_{L^2}^2$

$$\Rightarrow \sum_{n=1}^{N} \frac{\langle f, \varphi_n\rangle^2}{\|\varphi_n\|_{L^2}^2} \leq \|f\|_{L^2}^2 \quad \text{for } \underline{\underline{\text{any}}}\ N!$$

So we can take N as big as we want. In other words, we can take the limit as $N \to \infty$.

[The important point is to have $\|f\|_{L^2}^2$ finite, i.e. $f \in L^2$].

- Bessel's inequality: If $\{\varphi_n\}_{n\geq 1}$ is any orthogonal set of functions in $L^2(a,b)$ and $f \in L^2(a,b)$, then

1) $\sum_{n=1}^{\infty} c_n \varphi_n$ converges in L^2 (not necessarily to f).

$$\left(c_n = \frac{\langle f, \varphi_n\rangle}{\|\varphi_n\|_{L^2}^2} \right)$$

2) $$\left\| \sum_{n=1}^{\infty} c_n \varphi_n \right\|_{L^2}^2 = \sum_{n=1}^{\infty} c_n^2 \|\varphi_n\|_{L^2}^2 \leq \|f\|_{L^2}^2 .$$

φ_n are orthogonal

- Remark: 1) means that there may be functions in L^2 that cannot be perfectly approximated by the set $\{\varphi_n\}_{n\geq 1}$. That is, not every orthogonal set $\{\varphi_n\}_{n\geq 1}$ is a "basis" of L^2.

• **Definition**: The infinite orthogonal set $\{\varphi_n\}_{n\geq 1}$ is called complete in L^2 if any $f \in L^2$ is perfectly approximated (in L^2) by its Fourier Series. That is,

$$\{\varphi_n\}_{n\geq 1} \text{ complete} \iff \forall f \in L^2,\ f = \sum_{n=1}^{\infty} c_n \varphi_n \text{ in } L^2 .$$

$$\left(c_n = \frac{\langle f, \varphi_n \rangle}{\|\varphi_n\|_{L^2}^2} \right)$$

• **Theorem**: The orthogonal set $\{\varphi_n\}_{n\geq 1}$ is complete in L^2 if and only if

$$\Big\| \sum_{n=1}^{\infty} c_n \varphi_n \Big\|_{L^2} = \|f\|_{L^2} \text{ for any } f \in L^2 .$$

$$\left(\text{with } c_n = \frac{\langle f, \varphi_n \rangle}{\|\varphi_n\|_{L^2}^2} \right)$$

Proof: In pages -4-, -5- we found that

$$E_N = \Big\| f - \sum_{n=1}^{N} c_n \varphi_n \Big\|_{L^2}^2 = \|f\|_{L^2}^2 - \sum_{n=1}^{N} \frac{(\langle f, \varphi_n \rangle)^2}{\|\varphi_n\|_{L^2}^2} .$$

Notice that the last term is

$$\sum_{n=1}^{N} \frac{(\langle f, \varphi_n \rangle)^2}{\|\varphi_n\|_{L^2}^2} = \sum_{n=1}^{N} c_n^2 \|\varphi_n\|_{L^2}^2 = \Big\langle \sum_{n=1}^{N} c_n \varphi_n, \sum_{m=1}^{N} c_m \varphi_m \Big\rangle =$$

$$= \left\| \sum_{n=1}^{N} c_n e_n \right\|_{L^2}^2 .$$

By definition, $\sum_{n=1}^{\infty} c_n e_n = f$ in L^2 (or "converges to f in L^2")

if $\lim_{N\to\infty} \left\| \sum_{n=1}^{N} c_n e_n - f \right\|_{L^2} = 0$.

We can see above that

$\lim_{N\to\infty} E_N = \lim_{N\to\infty} \left\| f - \sum_{n=1}^{N} c_n e_n \right\|_{L^2}^2 = 0$ if and only if

$$\lim_{N\to\infty} \left(\|f\|_{L^2}^2 - \sum_{n=1}^{N} \frac{\langle f, e_n\rangle^2}{\|e_n\|_{L^2}^2} \right) = \lim_{N\to\infty} \left(\|f\|_{L^2}^2 - \left\| \sum_{n=1}^{N} c_n e_n \right\|_{L^2}^2 \right) = 0 .$$

In summary, we have shown that

$$f = \sum_{n=1}^{\infty} c_n e_n \text{ in } L^2 \iff \|f\|_{L^2}^2 = \left\| \sum_{n=1}^{\infty} c_n e_n \right\|_{L^2}^2 \quad // .$$

• <u>Remark</u>: The previous theorem gives a nice equality, very useful to compute (or estimate in real life) the norm of a function in L^2.

$\downarrow$

• Parseval's equality: If $\{e_n\}_{n\geq 1}$ is complete in L^2, then

$$\|f\|_{L^2}^2 = \sum_{n=1}^{\infty} c_n^2 \|e_n\|_{L^2}^2 \quad \text{for all } f \in L^2.$$

→ Notice that this equality allows to compute the norm of f from the coefficients of its expression in the "basis" $\{e_n\}_{n\geq 1}$.

(as we always used to do with column vectors).

• Remark: We have not proved Theorem 2 of Lecture 12.

↳ We've given characterizations of completeness in L^2.

↳ To prove it, one needs more analysis concepts (Hilbert spaces, dense subspaces)

[it's not hard with all that we have seen].

- All the previous results were stated for any orthogonal set $\{\varphi_n\}_{n\geq 1}$.

Going back to our problem $\left.\begin{array}{l} -\phi'' = \lambda\phi \\ \text{symmetric BCs} \end{array}\right\}$ [F]

we had Theorem 2, which now can be stated as:

<u>Theorem 2</u>: The eigenfunctions solving $-\phi'' = \lambda\phi$ with symmetric BCs form a complete set in L^2.

Therefore,

- <u>Corollary</u>: Let $\{\phi_n\}_{n\geq 1}$ be defined by [F] and $f \in L^2$.
  Then,

1) $f = \sum_{n=1}^{\infty} c_n \phi_n$ in L^2 with $c_n = \dfrac{\langle f, \phi_n\rangle}{\|\phi_n\|^2}$.

2) Parseval's equality holds: $\|f\|_{L^2}^2 = \sum_{n=1}^{\infty} c_n^2 \|\phi_n\|_{L^2}^2$.