MATH 241 LECTURE 13	: Higher Dimensional PDE I, Introduction.

Book Sections: 7.1, 7.2, 7.3

- So far in the course we have solved physical problems that can be modelled by a one-dimensional PDE, such as the wave or heat equation on an interval. (movement of a string, diffussion of heat in a rod).

- Mathematically, we have solved these (linear and homogeneous) PDEs using the method of separation of variables.
  - ↳ Write the unknow function as a product $(u(x,t) = \phi(x)G(t))$.
  - ↳ Transform the PDE into two ODEs.
  - ↳ Use the homogeneous BC to write a BVP.
    - ↳ Solve to find the λ_n and ϕ_n.

      [alsmost never explicitly ⤳ many times regular S-L]
  - ↳ Solve the other ODE with the given λ_n.
  - ↳ Use superposition: $u(x,t) = \sum_{n\geq 1} \phi_n(x) G_n(t)$.
  - ↳ Impose the initial conditions to determine constants.

- In higher dimensions, the steps are roughly the same. Conceptually, the ideas for the method of separation of variables work analogously. However, the technical details (i.e, solving each step) are harder.

→ Main differece: In 2d and 3d the geometry can be very complex!

2d example: Find the temperature distribution of the walls in a building.

3d example: Study the propagation of sound waves.

- We will limit ourselves to study the wave and heat equation (and the Laplace equation, which we know corresponds to the equilibrium solutions of the previous ones).

→ We will only be able to solve very particular cases, where the geometry is simple enough (rectangles, circles, spheres, ...).

Remark: The ideas work in complex geometries, but one need numerical method to implement them.

• Example: Heat Equation in 2d (general geometry)

PDE: $u_t(x,y,t) = k\left(u_{xx}(x,y,t) + u_{yy}(x,y,t)\right)$

BC: Any linear and homogeneous

↳ Dirichlet: $u(x,y,t) = 0$ for $(x,y) \in \partial\Omega$

↳ Neumann: $\vec{n} \cdot \nabla u(x,y,t) = 0$ for $(x,y) \in \partial\Omega$

↳ Robin: $\alpha u(x,y,t) + \beta \vec{n} \cdot \nabla u(x,y,t) = 0$ for $(x,y) \in \partial\Omega$

↳ (periodic)

IC: $u(x,y,0) = f(x,y)$.

Remark: The BC could be mixed
(Dirichlet on part of $\partial\Omega$, Neumann on some other part, ...).

Solution:

Separate time and space variables:

$u(x,y,t) = \phi(x,y)\, G(t)$,

and substitute into the PDE:

$$\phi(x,y)G'(t) = k\,G(t)\big(\phi_{xx}(x,y) + \phi_{yy}(x,y)\big) \Rightarrow$$

$$\frac{1}{k}\frac{G'(t)}{G(t)} = \frac{1}{\phi(x,y)}\big(\phi_{xx}(x,y) + \phi_{yy}(x,y)\big) = -\lambda$$

① $G'(t) = -\lambda k\, G(t)$

② $\phi_{xx}(x,y) + \phi_{yy}(x,y) = -\lambda\phi(x,y)$

<u>Remark</u>: 1) I chose $-\lambda$ so that the time ODE shows decay (as we expect physically). It is not crucial though.

2) ② can also be written as $\Delta\phi + \lambda\phi = 0$. (which reminds of $\phi'' + \lambda\phi = 0$ in 1d).

Now we have to write the BC in terms of ϕ. Let's assume for simplicity that we had Dirichlet BC along all the boundary:

$$\left.\begin{array}{l} \underset{\phi(x,y)G(t)}{\overset{u(x,y,t)}{\parallel}} = 0 \quad \text{for } (x,y)\in\partial\Omega \\ \qquad\qquad\qquad \text{for all } t\geq 0 \end{array}\right\} \Rightarrow \phi(x,y) = 0 \text{ for } (x,y)\in\partial\Omega. \quad [*]$$

Then, we solve ②

$$② \begin{rcases} \Delta u + \lambda \Phi = 0 \\ \Phi \equiv 0 \text{ on } \partial\Omega \end{rcases} \rightsquigarrow \begin{matrix} \lambda_n \\ \Phi_n(x,y) \end{matrix}$$

[We'll have a theory analogous to the S-L theory.]

(under certain assumptions on the domain Ω)

For each λ_n, we solve ①:

$$G_n(t) = e^{-\lambda_n k t}.$$

We construct the product solutions:

$$u_n(x,y,t) = A_n \Phi_n(x,y) e^{-\lambda_n k t},$$

and by superposition,

$$u(x,y,t) = \sum_{n\geq 1} A_n \Phi_n(x,y) e^{-\lambda_n k t}, \text{ with}$$

$$u(x,y,0) = \sum_{n\geq 1} A_n \Phi_n(x,y) = f(x,y).$$

[*] In most (real) problems we <u>cannot</u> separate further the variables x and y because of the geometry.

• Exercise: Wave Equation on a Rectangular Domain.

[Section 7.3]

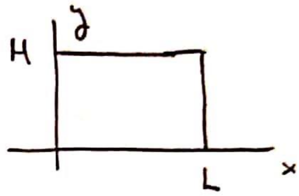
PDE: $u_{tt}(x,y,t) = c^2 \left(u_{xx}(x,y,t) + u_{yy}(x,y,t) \right),$

BCs: $u(0,y,t) = 0 = u(L,y,t), \quad 0<y<H$

$u(x,0,t) = 0 = u(x,H,t), \quad 0<x<L$

ICs: $u(x,y,0) = f(x,y)$

$u_t(x,y,0) = g(x,y)$



Solution:

Let $u(x,y,t) = \phi(x,y)\, h(t) \Rightarrow$

$$\frac{1}{c^2}\frac{h''(t)}{h(t)} = \frac{1}{\phi(x,y)}\left(\phi_{xx}(x,y) + \phi_{yy}(x,y) \right) = -\lambda$$

BCs: $u(0,y,t) = 0 \Rightarrow \phi(0,y) = 0$

$u(L,y,t) = 0 \Rightarrow \phi(L,y) = 0$, $0<y<H,$

$u(x,0,t) = 0 \Rightarrow \phi(x,0) = 0$

$u(x,H,t) = 0 \Rightarrow \phi(x,H) = 0$, $0<x<L.$

So we have

① $h''(t) = -\lambda c^2 h(t)$ } ODE

② $\Delta\phi + \lambda\phi = 0$

$\phi(0,y) = \phi(L,y) = 0$

$\phi(x,0) = \phi(x,H) = 0$

} BVP ← But now this is again a PDE !

In ②, the PDE and the BCs are linear and homogeneous. Thanks to the special geometry (rectangle), we can solve this eigenvalue problem by separation of variables.

Let $\phi(x,y) = F(x)G(y)$. The BCs can be written in terms of F and G:

$\phi(0,y) = F(0)G(y) = 0$ for $0<y<H \Rightarrow F(0) = 0$.

$\phi(L,y) = 0 \Rightarrow \quad \cdots \quad \Rightarrow F(L) = 0$.

$\phi(x,0) = 0 \Rightarrow \quad \cdots \quad \Rightarrow G(0) = 0$.

$\phi(x,H) = 0 \Rightarrow \quad \cdots \quad \Rightarrow G(H) = 0$.

The PDE in ② becomes

$$F''(x)G(y) + F(x)G''(y) = -\lambda F(x)G(y) \Rightarrow$$

$$\frac{F''(x)}{F(x)} = -\lambda - \frac{G''(y)}{G(y)} = -\mu \quad \Rightarrow$$

(2.1) $F''(x) = -\mu F(x)$ (2.2) $G''(x) = -(\lambda-\mu) G(y)$

$F(0) = F(L) = 0$ $G(0) = G(H) = 0$

Remark: Notice that we are writing

$$u(x,y,t) = \phi(x,y,t) = F(x)G(y)h(t)$$

$\uparrow$

$\left\{\right.$ This works in this problem (because simple geometry), but not in general.

Problems (2.1) and (2.2) are standard. We solve first (2.1) because it depends only on one parameter:

$$\mu_n = \left(\frac{n\pi}{L}\right)^2,\ n\geq 1 \text{ with } F_n(x) = \sin\left(\frac{n\pi x}{L}\right).$$

For each n, we solve (2.2) with this given value of μ:

$$\left.\begin{aligned} G''_n(x) &= -(\lambda_n-\mu_n)G_n(y) \\ G_n(0) &= G_n(H) = 0 \end{aligned}\right\} \rightarrow$$

$$\lambda_{n,m} - \mu_n = \left(\frac{m\pi}{H}\right)^2,\ m\geq 1, \text{ with}$$

$$\lambda_{n,m} = \underbrace{\left(\frac{n\pi}{L}\right)^2}_{\mu_n} + \left(\frac{m\pi}{H}\right)^2$$

$\Rightarrow$

$$G_{n,m}(y) = \sin\left(\frac{m\pi y}{H}\right).$$

- In summary, we have solved the 2d eigenvalue problem ②:

$$\lambda_{n,m} = \left(\frac{m\pi}{H}\right)^2 + \mu_n = \left(\frac{m\pi}{H}\right)^2 + \left(\frac{n\pi}{L}\right)^2, \quad \begin{matrix} n \geq 1, \\ m \geq 1, \end{matrix}$$

with

$$\phi_{n,m}(x,y) = F_n(x)\, G_{n,m}(y) = \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right).$$

- Now, back to ① with these values of $\lambda_{n,m}$:

① $h''_{n,m}(t) = -\lambda_{n,m} c^2 h_{n,m}(t) \rightarrow \{ \lambda_{n,m} > 0 \text{ for all } n,m \}$

$$h_{n,m}(t) = A_{n,m} \cos(c\sqrt{\lambda_{n,m}}\, t) + B_{n,m} \sin(c\sqrt{\lambda_{n,m}}\, t).$$

- Product solution:

$$u_{n,m}(x,y,t) = \phi_{n,m}(x,y)\, h_{n,m}(t).$$

- Superposition principle:

$$u(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} u_{n,m}(x,y,t) =$$

$$= \underbrace{\sum_{n=1}^{\infty} \sum_{m=1}^{\infty}} \left(A_{n,m} \cos(c\sqrt{\lambda_{n,m}}\, t) + B_{n,m} \sin(c\sqrt{\lambda_{n,m}}\, t)\right) \sin\left(\frac{m\pi y}{H}\right) \sin\left(\frac{n\pi x}{L}\right).$$

(we will always assume we can switch the order.)

• ICs :

$$u(x,y,0) = f(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{n,m} \sin\left(\frac{m\pi y}{H}\right)\sin\left(\frac{n\pi x}{L}\right),$$

$$u_t(x,y,0) = g(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c\sqrt{\lambda_{n,m}}\, B_{n,m} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right).$$

These are 2-d Fourier Series. We can find the coefficients $A_{n,m}$, $B_{n,m}$ by using orthogonality first in x and then in y (or vice versa).

Finding the coeff:

$$\int_0^L f(x,y)\sin\left(\frac{k\pi x}{L}\right)dx = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{n,m}\sin\left(\frac{m\pi y}{H}\right)\underbrace{\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{k\pi x}{L}\right)dx}_{=\begin{cases}0 & k\neq n\\ \frac{L}{2} & k=n\end{cases}} =$$

$$= \sum_{m=1}^{\infty} A_{k,m}\sin\left(\frac{m\pi y}{H}\right)\frac{L}{2}.$$

We repeat the procedure in y:

$$\int_0^H \left(\int_0^L f(x,y) \sin\left(\frac{k\pi x}{L}\right) dx \right) \sin\left(\frac{\ell \pi y}{H}\right) dy =$$

$$= \sum_{m=1}^{\infty} \frac{L}{2} A_{n,m} \int_0^H \sin\left(\frac{m\pi y}{H}\right) \sin\left(\frac{\ell \pi y}{H}\right) dy =$$

$$= \frac{L}{2}\frac{H}{2} A_{k,\ell} \Rightarrow$$

$$\Rightarrow \left\| A_{k,\ell} = \frac{4}{LH} \int_0^L \int_0^H f(x,y) \sin\left(\frac{k\pi x}{L}\right) \sin\left(\frac{\ell \pi y}{H}\right) dy\, dx \right. .$$

(Recall Fubini's rule to relate double and iterated integrals.)

Analogously, one can find that

$$c\sqrt{\lambda_{k,\ell}}\, B_{k,\ell} = \frac{4}{LH} \int_0^L \int_0^H g(x,y) \sin\left(\frac{k\pi x}{L}\right) \sin\left(\frac{\ell \pi y}{H}\right) dy\, dx .$$

<u>Remark</u>: Although this might seem complex, it is kind of surprising that we can completely describe how a rectangular membrane with fixed boundary vibrates in time with just one equation!

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