MATH 425 LECTURE 14: Fourier Series V

(Pointwise and Uniform Convergence: Proofs).

In this lecture we are going to prove the main part of Theorem 1 and Theorem 3 in Lecture 2.

These are important proofs because of the techniques that appear in them.

• Theorem 3: Let $f(x)$ be a C^1 function in $\mathbb{R}$, periodic with period 2π. Then, the classical Fourier series converges to $f(x)$ for all $x \in \mathbb{R}$.

Remark: • $f \in C^1$ means f and f' are continuous.

• The proof for f and f' piecewise continuous is very similar.

• The proof applies to functions define on $[a,b]$ upon rescaling and considering later the periodic extension.

→ The proof contains three main ideas or steps.

Proof:

Goal: Show that

$$f(x) = \sum_{n=0}^{\infty} (A_n \cos(nx) + B_n \sin(nx)) \quad \text{for all } x \in \mathbb{R}, \text{ where}$$

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\cos(ny)\,dy, \quad B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\sin(ny)\,dy,$$

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\,dy.$$

That is, we have to show that

$$\lim_{N\to\infty} \sum_{n=0}^{N} (A_n \cos(nx) + B_n \sin(nx)) = f(x) \quad \text{for all } x \in \mathbb{R},$$

which means (by definition) that for all $x \in \mathbb{R}$

$$\lim_{N\to\infty} |S_N(x) - f(x)| = 0, \text{ where}$$

$$S_N(x) = \sum_{n=0}^{N} (A_n \cos(nx) + B_n \sin(nx)) \quad \Big\| \text{ Partial sums.}$$

true for all $x \in \mathbb{R}$?

Steps :

1 | Dirichlet Kernel.

The first idea is to rewrite the partial sums as a convolution. We want to find a way to write $S_N(x)$ in terms of f but without the $\sum_{n=0}^{N}$ sum.

Let's substitute A_n, B_n by their formulas:

$$S_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\,dy + \sum_{n=1}^{N}\Big(\underbrace{\frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\cos(ny)\,dy}_{A_n}\Big)\cos(nx) +$$

$$+ \sum_{n=1}^{N}\Big(\underbrace{\frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\sin(ny)\,dy}_{B_n}\Big)\sin(nx) =$$

$$= \frac{1}{2\pi}\Big[\int_{-\pi}^{\pi} f(y)\,dy + \sum_{n=1}^{N} 2\int_{-\pi}^{\pi} f(y)\cos(ny)\cos(nx)\,dy +$$

$$+ \sum_{n=1}^{N} 2\int_{-\pi}^{\pi} f(y)\sin(ny)\sin(nx)\,dy\Big] =$$

$$= \frac{1}{2\pi}\Big[\int_{-\pi}^{\pi} f(y)\,dy + 2\sum_{n=1}^{N}\int_{-\pi}^{\pi} f(y)\big(\underbrace{\cos(ny)\cos(nx) + \sin(ny)\sin(nx)}_{=\cos(n(x-y))}\big)\,dy\Big] =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(1 + 2\sum_{n=1}^{N}\cos(n(x-y))\right)f(y)\,dy.$$

Denote

$$\boxed{K_N(x) = 1 + 2\sum_{n=1}^{N}\cos(nx)}$$

DIRICHLET KERNEL.

Then, we've shown that

$$S_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x-y)f(y)\,dy := \frac{1}{2\pi}\overbrace{(K_N * f)}^{\text{convolution}}(x).$$

2| Dirichlet Kernel: "Closed form"

<u>Claim</u>: $$\boxed{K_N(x) = \frac{\sin\left((2N+1)\frac{x}{2}\right)}{\sin(x/2)}}$$

Π Check: Recall that $\cos(nx) = \dfrac{e^{inx} + e^{-inx}}{2}$. Then,

$$1 + 2\sum_{n=1}^{N}\cos(nx) = \underbrace{1}_{= e^{i0x}} + \sum_{n=1}^{N}\left(e^{inx} + e^{-inx}\right) = \sum_{n=-N}^{N} e^{inx}.$$

This is a geometric series, easy to sum. Indeed,

$$\left.\begin{aligned} K_N(x) &= e^{-iNx} + e^{-i(N-1)x} + \dots + 1 + \dots + e^{i(N-1)x} + e^{iNx}, \\ e^{ix} K_N(x) &= \qquad\qquad e^{-i(N-1)x} + \dots + 1 + \dots + e^{iNx} + e^{i(N+1)x} \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow K_N(x)(1-e^{ix}) = e^{-iNx} - e^{i(N+1)x} \Rightarrow$$

$$\Rightarrow K_N(x) = \frac{e^{-iNx} - e^{i(N+1)x}}{1-e^{ix}} \underset{\uparrow \cdot e^{-ix/2}}{=} \frac{e^{-i(N+\frac{1}{2})x} - e^{i(N+\frac{1}{2})x}}{e^{-ix/2} - e^{ix/2}} = \frac{\sin((N+\frac{1}{2})x)}{\sin(x/2)} \,/\!/.$$

$$\left(\sin z = \frac{e^{iz} - e^{-iz}}{2i} \right)$$

- An important property about the Dirichlet Kernel is that

1) $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x)\,dx = 1$ for any $N \geq 1$ (prove it!) ← exercise.

[2) $\lim_{N\to\infty} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |K_N(x)|\,dx \right) = +\infty$ (try to prove)
↳ if interested, Google.
(not needed for our purposes). (many more things about Dirichlet Kernel).]

3 Using L^2 theory.

We wanted to show $\lim_{N\to\infty} |S_N(x) - f(x)| = 0$ (for all $x \in \mathbb{T}$).

Now we know that we can write

$$S_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x-y) f(y)\, dy.$$

Or, changing variables $y \leftarrow x-y$,

$$S_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(y) f(x-y)\, dy$$

This is the commutative property for convolutions:

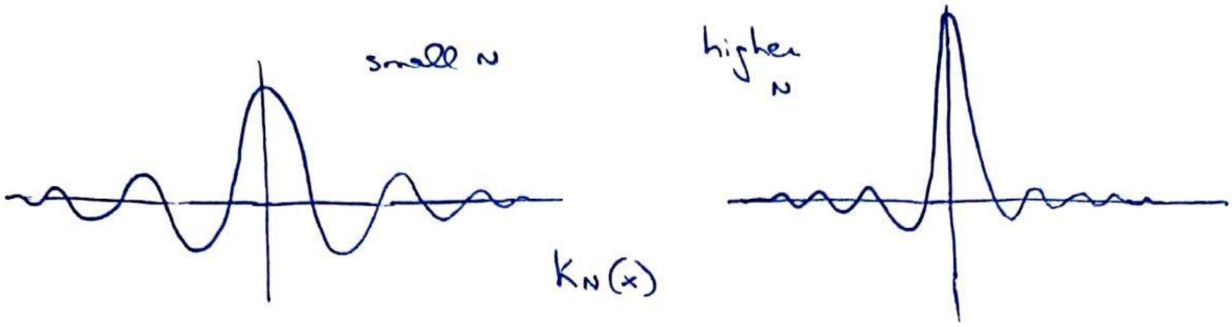
$$(f*g)(x) = \int_{-\pi}^{\pi} f(x-y) g(y)\, dy = \int_{-\pi}^{\pi} f(y) g(x-y)\, dy = (g*f)(x)$$

Because $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(y)\, dy = 1$, we can write

$$S_N(x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(y)\big(f(x-y) - f(x)\big)\, dy$$

This is a standard trick to use when one deals with a kernel. Notice we have introduce certain "cancellation" on the numerator, needed to "kill" the singularity in the denominator of $K_N(y)$ when $y \approx 0$.

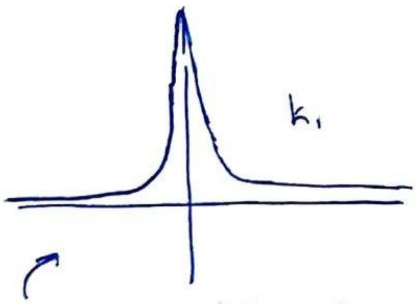
It might help thinking about K_N. It looks like

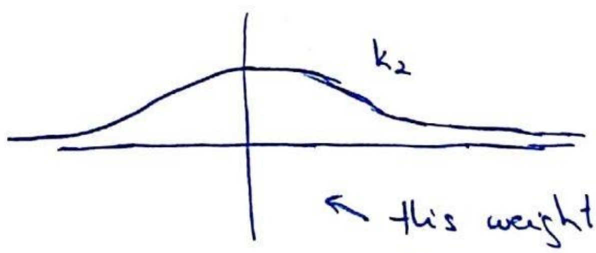


As $N \to +\infty$, the Dirichlet kernel "approaches the Dirac delta distribution".

[Remember the heat kernel, the intuition about convolutions, and how the solution approaches the initial data as $t \to 0^+$.]

$$\int_{-\infty}^{\infty} k(x-y) f(y)\, dy \approx \underbrace{\text{weighted}}_{\text{the profile of } k \text{ gives the weight.}} \text{ average of } f \text{ at } x.$$





with this weight, only the values of f near x matter: that is,

$(k_1 * f)(x) \approx f(x)$

this weight mixes everything:

$(k_2 * f)(x) \approx$ mean value of f

(we "lose" the information near x).

- We want to prove that $S_N(x) \to f(x)$ as $N \to \infty$.

[This seems likely true now that we understand that $S_N(x)$ tends to a Dirac delta as $N \to \infty$.]

Rigorously, we want to show that

$$S_N(x) - f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} K_N(y)\big(f(x-y) - f(x)\big)\,dy \to 0 \quad \text{as } N \to \infty .$$

→ The idea is to use the smoothness of f (it's piecewise differentiable) to get rid of the singularity in $K_N(y)$:

$$S_N(x) - f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin\left(\left(N+\tfrac{1}{2}\right)y\right) \underbrace{\frac{f(x-y)-f(y)}{\sin(y/2)}}_{\approx \text{ a derivative of } f} \,dy$$

independent of N.

(this is why we need to have the "cancellation" $f(x-y) - f(x)$)

<u>Question:</u> Is $\displaystyle\lim_{N\to\infty} \int_{-\pi}^{\pi} \sin\left(\left(N+\tfrac{1}{2}\right)y\right) g(y)\,dy = 0$ for any g?

Here is where our knowledge of the L^2 theory comes into play. We can write the above integral as an inner product in L^2:

- $\int_{-\pi}^{\pi} \sin((N+\frac{1}{2})y)\, g(y)\, dy = < \sin((N+\frac{1}{2})x), g(x) >$.

Why should we expect this to go to zero as $N \to +\infty$?

↳ Think of $\{\sin((n+\frac{1}{2})x)\}_{n\geq 1}$ as a "basis" of L^2. Then, $< \sin((N+\frac{1}{2})x), g(x) >$ is the N^{th} component of g on that basis. We know that if $g \in L^2$, then it can be well approximated by any basis of L^2, so its components are smaller and smaller as $N \to +\infty$.

- Assume for now that $\{\sin((n+\frac{1}{2})x)\}_{n\geq 1}$ is indeed an orthogonal set in L^2 (we don't really need it to be an actual "basis", i.e, complete).

- Assume also that $g \in L^2$.

Then, Bessel's inequality guarantees that:

$$\sum_{n=1}^{\infty} \frac{(< \sin((n+\frac{1}{2})x), g(x) >)^2}{\| \sin((n+\frac{1}{2})x)\|_{L^2}^2} \leq \|g\|_{L^2}^2 < +\infty .$$

Therefore, since the series (of numbers) is convergent, its terms must tend to zero: $\left(\| \sin((n+\frac{1}{2})x) \|_{L^2}^2 = \pi \right)$

$$< \sin((n+\tfrac{1}{2})x), g(x) > \to 0 \quad \text{as} \quad n \to +\infty$$

- Let's check the two assumptions:

1) $\{ \sin((n+\frac{1}{2})x) \}_{n \geq 1}$ is an orthogonal set in L^2:

It can be checked (exercise) that these are the eigenfunctions of
$$\left. \begin{array}{l} \phi'' + \lambda\phi = 0, \quad 0 < x < \pi \\ \phi(0) = 0 = \phi'(\pi) \end{array} \right\}$$

Remark: So they are even complete in L^2. We could have used Parseval's equality instead of Bessel's ineq.

2) $g \in L^2$

Remember that we were trying to prove if

$$S_N(x) - f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin((N+\tfrac{1}{2})y) \frac{f(x-y) - f(y)}{\sin(y/2)} \, dy \to 0 \quad \text{as } N \to \infty.$$

So for each x, denote
$$g_{(x)}(y) = \frac{f(x-y) - f(y)}{\sin(y/2)}.$$

Is $g_{(x)}(y)$ in L^2 (for any fixed x)?

↳ Yes, thanks to the differentiability of f:

$$\|g_{(x)}\|_{L^2}^2 = \int_{-\pi}^{\pi} \frac{|f(x-y)-f(y)|^2}{|\sin(y/2)|^2}\,dy = \int_{-\pi}^{\pi} \frac{|f(x-y)-f(y)|^2}{|y/2|^2}\,\frac{(y/2)^2}{\sin^2(y/2)}\,dy \leq$$

$$\leq 4 \max_{[-\pi,\pi]} |f'(x)| \int_{-\pi}^{\pi} \frac{(y/2)^2}{\sin^2(y/2)}\,dy < +\infty .$$

(near $y \approx 0$, $\sin(y/2) \approx y/2$)

(mean-value theorem)

This concludes the proof.

→ Summary:

$$S_N(x) = \sum_{n=0}^{N} \left(A_n \cos(nx) + B_n \sin(nx)\right) \xrightarrow{N\to+\infty} f(x) \quad \text{for all } x \in \mathbb{R}\,?$$

↳ $S_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(y)\, f(x-y)\,dy$

↳ $$S_N(x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(y)\left(f(x-y)-f(y)\right)dy = \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin\left(\left(N+\tfrac{1}{2}\right)y\right)\frac{f(x-y)-f(y)}{\sin(y/2)}\,dy}_{} .$$

$$= \Big\langle \underbrace{\sin\left(\left(N+\tfrac{1}{2}\right)y\right)}_{\text{orthogonal set in } L^2},\ \underbrace{\frac{f(x-y)-f(y)}{\sin(y/2)}}_{\in L^2} \Big\rangle$$

Theorem 1: (Uniform Convergence).

f continuous, 2π-periodic
f' piecewise continuous $\Bigg\} \Rightarrow$ The (classical) Fourier series of f converges uniformly. (in $\mathbb{R}$).

Remark: We don't prove the general case (eigenfunctions ϕ_n) because the proof we are using needs to assume the pointwise convergence. Notice that in the proof for pointwise convergence we used crucially the form of the eigenfunctions (to compute the Dirichlet kernel).

Proof:

Goal: $\lim\limits_{N\to+\infty} \sup\limits_{x\in\mathbb{R}} |f(x) - S_N(x)| = 0$? $\left(\begin{array}{c} S_N(x) \text{ same} \\ \text{as before} \end{array}\right)$.

→ The conditions on f are enough to guarantee the pointwise convergence of $S_N(x)$ to $f(x)$ for all $x\in\mathbb{R}$.

→ The idea is to use Weierstrass' test: For that, we need that the Fourier coefficients decay fast enough.

- Denote A_n, B_n the Fourier coeff. of $f(x)$, and A_n', B_n' the coeff. of $f'(x)$. Using integration by parts we can relate them:

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{n\pi} \cancel{f(x)\sin(nx)\Big]_{-\pi}^{\pi}} - \frac{1}{n\pi}\int_{-\pi}^{\pi} f'(x)\sin(nx)\,dx =$$

$$= \frac{-1}{n} B_n', \quad (n \geq 1)$$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \dots = \frac{1}{n} A_n'.$$

- Because $f'(x)$ is piecewice continuous, it is in $L^2([-\pi,\pi])$. Thus, Bessel's inequality applies:

$$\sum_{n=1}^{\infty}\left(|A_n'|^2 + |B_n'|^2\right) < \infty$$

$\sum_{n=1}^{\infty} c_n^2 \|\varphi_n\|_{L^2}^2 \leq \|f\|_{L^2}^2$ for

$f \sim \sum_{n=1}^{\infty} c_n \varphi_n$

Here, $\phi_n \rightsquigarrow \cos(nx), \sin(nx)$

$c_n \rightsquigarrow A_n', B_n'$

Therefore,

$$\sum_{n=1}^{\infty}\left(|A_n\cos(nx)|+|B_n\sin(nx)|\right)\leq\sum_{n=1}^{\infty}\left(|A_n|+|B_n|\right)=$$

$$=\sum_{n=1}^{\infty}\frac{1}{n}\left(|B_n'|+|A_n'|\right)\leq\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{1/2}\left(\sum_{n=1}^{\infty}\left(|B_n'|+|A_n'|\right)^2\right)^{1/2}\leq$$

Schwarz's ineq. $\sum ab\leq(\sum a^2)^{1/2}(\sum b^2)^{1/2}$.

$$\leq\frac{\pi}{\sqrt{6}}\left(\sum_{n=1}^{\infty}2\left(|A_n'|^2+|B_n'|^2\right)\right)^{1/2}<\infty.$$

$(a+b)^2\leq 2(a^2+b^2)$

→ The important point was the decay (and of course that $f'\in L^2$)

$$|A_n|=\frac{1}{n}|B_n'|$$
$$|B_n|=\frac{1}{n}|A_n'|.$$

- Finally, we can apply Weierstrass' tests:

$$S_N(x)=\sum_{n=0}^{\infty}\left(A_n\cos(nx)+B_n\sin(nx)\right)$$

$$|A_n\cos(nx)+B_n\sin(nx)|\leq|A_n|+|B_n|$$

$$\sum_{n=0}^{\infty}\left(|A_n|+|B_n|\right)<+\infty$$

$\Rightarrow$ $S_N(x)$ converges uniformly to $f(x)$. ✓