MATH 241 LECTURE 14 : Higher Dimensional PDEs II (Helmholtz Equation and Generalities)

- Last lectures: We know how to

1) Solve 1-d linear and homogeneous PDEs with constant coefficients, such as the heat and wave equations (with appropriate homogeneous BCs).

2) Calculate the "energy" of the solutions for the 1d heat and wave equations.
Moreover, we understand the relation of the solutions to the Laplace equation with those of the heat equation through the conservation of energy (the former corresponds to equilibrium solutions of the latter).

3) "Solve" more general linear and homogeneous PDEs, such as heat and wave equations with non-constant coefficients, in terms of the eigenfunctions (S-L Theory).

4) Solve the heat and wave equation (and similar PDEs) with constant coefficients in rectangular domains (2d rectangle, 3d box).

- In this lecture, we first want to give some analogous results to those of Sturm-Liouville theory for 2d and 3d.

  ↳ And use them to deduce qualitative properties of the solutions, such as growth in time.
  (typically related to determining if $\lambda \geq 0$, $\lambda > 0$, ...).

- We also want to briefly study how to compute the "energy" in higher dimensional problems (and its relation to the Laplace equation: remember the problems for the Laplace equation on a rectangle and the necessity of certain compatibility condition when all the BCs were of Neumann type).

→ Book Sections: 7.4, 7.6.2.

[Section 7.4] The Helmholtz Equation.

- In 1-d, our typical eigenvalue problem was

$$\left.\begin{array}{l}\phi'' + \lambda\phi = 0\\ + \text{ BC}\end{array}\right\}$$

Later, Sturm-Liouville theory allowed us to consider much more general eigenvalue problems,

$$\left.\begin{array}{l}\frac{d}{dx}\left(p(x)\phi'(x)\right) + q(x)\phi(x) + \lambda\sigma(x)\phi(x) = 0\\ \quad + \text{ BC.}\end{array}\right\}$$

because under certain conditions (regular S-L problems) all the important properties were still true.

- In 2-d, we found in last lecture that, for both the heat and wave equation, the eigenvalue problem is

$$\left.\begin{array}{l}\Delta\phi + \lambda\phi = 0\\ + \text{ BCs}\end{array}\right\}$$

Although we could generalise this to $\nabla\cdot(p\nabla\phi) + q\phi + \lambda\sigma\phi = 0$, we will limit ourselves to the simpler case above:

$$\boxed{\Delta \phi + \lambda \phi = 0}$$ HELMHOLTZ EQUATION.

- Even for this simpler case with Dirichlet, Neumann, or Robin BCs,

$$\left.\begin{aligned} &\Delta \phi + \lambda \phi = 0 \\ &\alpha \phi + \beta n \cdot \nabla \phi = 0 \end{aligned}\right\} \text{[HBVP]} \quad \text{(2d and 3d)}$$

most of the times we cannot compute λ and ϕ explicitly.

However, if the domain is smooth (the boundary has a well-defined, smooth tangent vector everywhere), the following properties hold for [HBVP]:

1] All λ are real.

2] There exists a smallest eigenvalue λ_1, an infinite number of eigenvalues, and no largest eigenvalue. ($\lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots$, $\lambda_n \to \infty$ as $n \to \infty$).

3] [*]

4] The eigenfunctions form a complete set: any piecewise smooth function $f(x,y)$ (or $f(x,y,z)$ in 3d) can be written as a generalized Fourier series

$$f(x,y) \sim \sum_{\lambda} A_\lambda \phi_\lambda(x,y),$$

with the coefficients A_λ appropriately chosen. The sum $\sum_{\lambda}$ indicates summation over all the eigenvalues (it will typically be a double infinite sum in 2d, and a triple one in 3d).

5] Eigenfunctions corresponding to different eigenvalues are orthogonal over the domain (with weight $\sigma = 1$),

$$\int_{\Omega} \phi_{\lambda_1} \phi_{\lambda_2} \, dA = 0 \quad \text{if } \lambda_1 \neq \lambda_2$$

$\left(\begin{array}{c} dA = dx\,dy \\ \text{typically in 2d} \end{array} \right)$.

$\left(\int_{\Omega} \begin{array}{l} \to \text{double integral in 2d} \\ \to \text{triple integral in 3d} \end{array} \right)$

[**]

6] Rayleigh quotient:

$$\boxed{\lambda = \frac{-\int_{\partial\Omega} \phi \, \vec{n} \cdot \nabla \phi \, dS + \int_{\Omega} |\nabla \phi|^2 \, dA}{\int_{\Omega} \phi^2 \, dA}}$$

[*] For each eigenvalue there might be many eigenfunctions (as opposed to 1-d).

[**] If an eigenvalue has more than one (independent) eigenfunction (i.e. eigenfunctions that are not just multiples of each other), one can use the Gram-Schmidt algorithm to make the eigenfunctions orthogonal.

Remarks:

1) Check all these properties in the wave equation exercise done in Lecture 13.

2) The Gram-Schmidt algorithm is the same here than for usual column vectors (see Homework 1).

↳ If interested, see Appendix in 7.5 (Book). [Optional]

• Exercise: [Section 7.6.2]

Consider the two-dimensional eigenvalue problem

$$\left.\begin{aligned} &\Delta\phi + \lambda\phi = 0 \quad \text{in } \Omega\subset\mathbb{R}^2 \\ &\phi = 0 \quad \text{on } \partial\Omega \end{aligned}\right\}, \qquad (\Omega \text{ connected domain})$$

↳ I.e., "one piece"

a) Prove that $\lambda \geq 0$.

b) Is $\lambda = 0$ an eigenvalue? If so, what is an eigenfunction corresponding to $\lambda = 0$?

Solution:

a) We cannot compute the values of λ (because we don't know the domain Ω).

Let's use the Rayleigh quotient:

$$\lambda = \frac{-\cancel{\int_{\partial\Omega} \phi\,\vec{n}\cdot\nabla\phi\, dl} + \int_{\Omega} |\nabla\phi|^2\, dx\, dy}{\int_{\Omega} \phi^2\, dx\, dy} = \frac{\int_{\Omega} |\nabla\phi|^2\, dx\, dy}{\int_{\Omega} \phi^2\, dx\, dy} \geq 0 \,/\!/.$$

b) Let's assume that $\lambda = 0$. Then,

$$0 = \frac{\int_{\Omega} |\nabla \phi|^2 \, dx dy}{\int_{\Omega} \phi^2 \, dx dy} \Rightarrow \int_{\Omega} |\nabla \phi|^2 \, dx dy = 0 \Rightarrow$$

$$\Rightarrow |\nabla \phi(x,y)| = 0 \text{ for all } (x,y) \in \Omega \Rightarrow$$

$$\Rightarrow \nabla \phi(x,y) = 0 \text{ for all } (x,y) \in \Omega \Rightarrow \phi(x,y) = C$$

[(*) Here it is where one needs the condition that Ω is connected]
(we will not worry about these details)

Finally, since $\phi = 0$ on $\partial\Omega$ and ϕ is a constant, we obtain that $\lambda = 0$ is not an eigenvalue (because $\phi \equiv 0$ cannot be an eigenfunction).

• <u>Remark</u>: Notice that the previous eigenvalue problem appears when solving the heat or wave equation with Dirichlet conditions.

↳ Interpretation:

- Heat Equation: The time ODE will have an exponential solution,

$$G'(t) = -\lambda k G(t) \rightsquigarrow G(t) = e^{-\lambda k t},$$

which shows decay to zero since $\lambda > 0$.

This is what we should expect if we are studying the heat equation in a domain Ω without heat sources and fixed temperature on $\partial\Omega$ equal to zero.

- Wave Equation: the time ODE will give

$$G''(t) = -c^2 \lambda G(t) \rightsquigarrow G(t) = c_1 \cos(c\sqrt{\lambda}\, t) + c_2 \sin(c\sqrt{\lambda}\, t)$$

thanks to the fact that $\lambda > 0$.

That is, if the boundary of the membrane is fixed, the membrane will oscillate forever (notice that we are not considering damping). We will later check that $u_{tt} = c^2 \Delta u$ preserves indeed the energy.

• <u>Remark</u>: We have also prove that the solution to the Laplace equation with Dirichlet BC,

$$\left.\begin{array}{ll} \Delta u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{array}\right\} \quad \text{is} \quad u \equiv 0.$$

This corresponds with the fact this is the equilibrium solution (time-indepent) to the heat equation

$$\left.\begin{array}{ll} u_t = k\Delta u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{array}\right\},$$

and that $\lim_{t\to\infty} u(x, y, t) = 0$.

$u \equiv 0$ is also a time-independent solution to the wave equation with fixed boundary. However, solutions to this equation don't go to a time-independent equilibrium.

- **Exercise:** Show that $\lambda \geq 0$ for

$$\left.\begin{aligned} \Delta\phi + \lambda\phi &= 0 \quad \text{in } \Omega \subset \mathbb{R}^2 \\ n \cdot \nabla\phi &= 0 \quad \text{on } \partial\Omega \end{aligned}\right\}$$

and that $\lambda = 0$ is an eigenvalue.

Solution:

Repeating the same steps, we obtain from the Rayleigh quotient that

$$\lambda = \frac{\int_\Omega |\nabla\phi|^2 \, dx\,dy}{\int_\Omega \phi^2 \, dx\,dy} \geq 0 .$$

If $\lambda = 0$, then $\int_\Omega |\nabla\phi|^2 \, dx\,dy = 0 \Rightarrow \nabla\phi(x,y) = 0$ in $\Omega \Rightarrow$

$\Rightarrow \phi = c$.

This is not a contradiction. Moreover, we can check that for $\lambda = 0$, $\phi(x,y) = c \neq 0$ solves the PDE and the BCs, thus it is an eigenfunction. Therefore, $\lambda = 0$ is an eigenvalue.

• **Remark:** Consider the heat equation with insulation BC,

$$\left.\begin{array}{ll} u_t = k\,\Delta u & \text{in } \Omega \subset \mathbb{R}^2 \\ n\cdot\nabla u = 0 & \text{on } \partial\Omega \end{array}\right\}$$

and initial temperature distribution $u(x,y,0) = f(x,y)$.

Which is the equilibrium solution to this problem? That is, towards which temperature distribution the solution $u(x,y,t)$ converges as $t\to\infty$?

→ Notice that we can answer this question for any Ω, we don't need to find $u(x,y,t)$:

After separation of variables, we will find

$u(x,y,t) = \sum_{\lambda} A_\lambda e^{-\lambda k t}\phi_\lambda(x,y)$, where

$\phi_\lambda(x,y)$ is determined by

$$\left.\begin{array}{ll} \Delta\phi + \lambda\phi = 0 & \text{in } \Omega \\ n\cdot\nabla\phi = 0 & \text{on } \partial\Omega \end{array}\right\} \rightsquigarrow \text{so } \lambda \geq 0 \text{ (with } \lambda = 0 \text{ included).}$$

Therefore, $\lim_{t\to\infty} u(x,y,t) = A_0$.

As in 1-d, we can find this constant by relating the energy of the initial state with that of the final (equilibrium) solution. Mathematically, this conservation of energy can be deduced by integrating the PDE:

$$\int_\Omega u_t(x,y,t)\,dx\,dy = \int_\Omega k\,\Delta u(x,y,t)\,dx\,dy =$$

integration by parts

$$= k\int_{\partial\Omega} n\cdot\nabla u\,dl = 0$$

$\boxed{\text{Integration by parts}}$:

$$\int_\Omega f\,\partial_{x_i} g = -\int_\Omega \partial_{x_i} f\, g + \int_{\partial\Omega} f\,g\,n_i ,$$

where $\Omega \equiv$ domain in 1,2, or 3d,

$\partial_{x_i} \equiv$ partial derivative in the i variable,

$n_i \equiv$ i^{th} component of the unit outward normal to $\partial\Omega$

- Particular case:

$$\int_\Omega f\,\Delta g = -\int_\Omega \nabla f\cdot\nabla g + \int_{\partial\Omega} f\,n\cdot\nabla g$$

Therefore,

$$\frac{d}{dt}\int_{\Omega} u(x,y,t)\,dx\,dy = 0 \rightarrow \int_{\Omega} u(x,y,t)\,dx\,dy = \int_{\Omega} u(x,y,0)\,dx\,dy$$

In the equilibrium situation ($t \rightarrow +\infty$), denote the temperature distribution by $u_\infty(x,y)$. Then,

$$\int_{\Omega} u_\infty(x,y)\,dx\,dy = \int_{\Omega} f(x,y)\,dx\,dy \Rightarrow \{u_\infty(x,y) = A_0\}$$

$$\Rightarrow \quad A_0 = \underbrace{\frac{1}{|\Omega|}\int_{\Omega} f(x,y)\,dx\,dy}_{\text{mean value of the initial temperature.}} \quad // \qquad [\,|\Omega| = \text{area of } \Omega\,].$$

- **Remark:** (optional) The integration by parts formula can easily be used to show the Rayleigh quotient formula:

$$\Delta\phi + \lambda\phi = 0 \Rightarrow \phi\,\Delta\phi + \lambda\phi^2 = 0 \Rightarrow \int_{\Omega}\phi\,\Delta\phi\,dA = -\lambda\int_{\Omega}\phi^2\,dA.$$

$$\Rightarrow -\int_{\Omega}\nabla\phi\cdot\nabla\phi\,dA + \int_{\partial\Omega}\phi\,n\cdot\nabla\phi\,dS = -\lambda\int_{\Omega}\phi^2\,dA \Rightarrow \lambda = \frac{-\int_{\partial\Omega}\phi\,n\cdot\nabla\phi\,dS + \int_{\Omega}|\nabla\phi|^2\,dA}{\int_{\Omega}\phi^2\,dA} \quad //.$$

• More on integration by parts and energy:

• Example: Consider Laplace's equation $\Delta u = 0$ in a 3d region, where $u(x,y,z)$ is the (time-independent) temperature.

Suppose that the heat flux is given on the boundary,

$$\left.\begin{array}{ll} \Delta u = 0 & \text{in } \Omega \subset \mathbb{R}^3 \\ n \cdot \nabla u = f & \text{on } \partial\Omega \end{array}\right\} \quad (\partial\Omega \text{ is a closed surface})$$

[↳ Think of Ω as a ball and $\partial\Omega$ the sphere surface]

a) Explain physically why $\int_{\partial\Omega} f \, dS = 0$ is a necessary condition for a solution to exists.

$f(x,y,z)$ is the heat flux across the boundary $\partial\Omega$. For an equilibrium to exist, the total energy should be constant in time. Since there are not heat sources, the net flux across the surface $\partial\Omega$ must be zero.

b) Show as mathematically.

If u is a solution, then

$$\int_\Omega \Delta u \, dV = 0 \xRightarrow{\text{int. by parts}} \int_{\partial\Omega} n\cdot\nabla u \, dS = 0. \text{ That is,}$$

if u is a solution, it must hold that $\int_{\partial\Omega} g \, dS = 0$.

[Thus, if $\int_{\partial\Omega} g \, dS \neq 0$, then there is no solution].

- <u>Example</u>: Suppose u solves $u_t = \Delta u$ in a bounded domain $B \subset \mathbb{R}^3$ subject to

$$\left.\begin{aligned} &u(x,y,z,0) = f(x,y,z) \text{ in } B, \\ &n\cdot\nabla u = g \text{ on } \partial B \text{ for } t \geq 0 \end{aligned}\right\}$$

Let $E(t)$ be the total energy, $E(t) = \int_B u(x,y,z,t)\,dx\,dy\,dz$.

Find the total energy at $t = 100$.

$$E'(t) = \int_B u_t(x,y,z,t)\,dx\,dy\,dz = \int_B \Delta u(x,y,z,t)\,dx\,dy\,dz =$$

$$= \int_{\partial B} n\cdot\nabla u \, dS = \underbrace{\int_{\partial B} g \, dS}_{\text{(this is a number)}} .$$

Thus,

$$E(t) = \left(\int_{\partial B} g\, dS\right)t + E(0) = \left(\int_{\partial B} g\, dS\right)t + \underbrace{\int_B f(x,y,z)\,dx\,dy\,dz}_{\text{this is a number.}}$$

Therefore,

$$E(100) = 100\,A + B, \text{ with } A = \int_{\partial B} g\, dS,\ B = \int_B f\, dV .$$