MATH 241	:	Higher Dimensional PDEs III
LECTURE 16		(Modified Bessel's Functions)

This Lecture is another example of problems in higher dimensions with simple enough geometry to completely separate all the variables: we will be working in a cylinder.

- In solving the heat/wave equation in a circle (more exactly, in solving the Helmholtz equation, that is, the spatial part), we found the Bessel's functions.

Now we will find again these functions, as well as new ones: the modified Bessel's functions.

- Problem: Laplace Equation in a cylinder.

PDE: $\Delta u = 0$, $-\pi < \theta < \pi, \ 0 < r < R, \ 0 < z < H,$

BCs: $u = \alpha$ on $z = H$ (top)

$\vec{n} \cdot \nabla u = \beta$ on $z = 0$ (bottom)

$u = \gamma$ on $r = R$ (lateral)

(α, β, γ given functions)

• Solution: First, write the problem in cylindrical coordinates,

$$\text{PDE: } \Delta u(r,\theta,z) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0,$$

BCs:

$$\left.\begin{array}{ll} u(r,\theta,H) = \alpha(r,\theta), & \\ -u_z(r,\theta,0) = \beta(r,\theta), & \text{the procedure is similar with other BCs.} \\ u(R,\theta,z) = \gamma(\theta,z) & \end{array}\right\} [L]$$

→ As in 2d (Laplace eq. in a rectangle, circle, ...), we cannot use separation of variables directly with more than one nonhomogeneous BC.

(↳ we are not able to separate variables in the non-homog. BCs, and we would not find enough eigenfunction expansions).

We need to use the linearity of the problem to decompose the solution:

$u = u_1 + u_2 + u_3$, where u_i solves the Laplace equation with only one nonhomogeneous BC. In particular,

u_1, u_2, u_3 are the solutions to:

① $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_1}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_1}{\partial \theta^2}+\frac{\partial^2 u_1}{\partial z^2}=0, \quad 0<r<R,\ -\pi<\theta<\pi,\ 0<z<H,$$

BCs:
$$u_1(r,\theta,H)=\alpha(r,\theta),$$
$$-\frac{\partial u_1}{\partial z}(r,\theta,0)=0,$$
$$u_1(R,\theta,z)=0.$$

② $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_2}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_2}{\partial \theta^2}+\frac{\partial^2 u_2}{\partial z^2}=0, \quad 0<r<R,\ -\pi<\theta<\pi,\ 0<z<H$$

BCs:
$$u_2(r,\theta,H)=0,$$
$$-\frac{\partial u_2}{\partial z}(r,\theta,0)=\beta(r,\theta),$$
$$u_2(R,\theta,z)=0$$

③ $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_3}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_3}{\partial \theta^2}+\frac{\partial^2 u_3}{\partial z^2}=0, \quad 0<r<R,\ -\pi<\theta<\pi,\ 0<z<H$$

BCs
$$u_3(r,\theta,H)=0,$$
$$-\frac{\partial u_3}{\partial z}(r,\theta,0)=0,$$
$$u_3(R,\theta,z)=\gamma(\theta,z).$$

<u>Exercise</u>: Check that if u_1, u_2, u_3 solve ①, ②, ③, respectively, then $u=u_1+u_2+u_3$ solves [L].

- Now we must solve each of these three problems separately.

- Solving ① : Laplace eq. in cylinder with
  - given temperature on top,
  - insulation on bottom.
  - zero temperature on lateral sides.

The variable z seems to play a role different than r, θ. We separate thus as follows:

$u_1(r,\theta,z) = \phi(r,\theta)\, h(z)$, to obtain

$h(z)\,\Delta_2\phi(r,\theta) + \phi(r,\theta)\, h''(z) = 0$, where we are using the notation $\Delta_2 \equiv$ 2d Laplacian

(i.e., in polar coord., $\Delta_2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$).

Then,

$$-\frac{\Delta_2\phi}{\phi} = \frac{h''(z)}{h(z)} = \lambda .$$

- BCs : $-\frac{\partial u_1}{\partial z}(r,\theta,0) = 0 = -\phi(r,\theta)\,h'(0) = 0$ for all $r,\theta \Rightarrow$

$\Rightarrow h'(0) = 0$

$u_1(R,\theta,z) = \phi(R,\theta)\,h(z) = 0$ for all $z \rightarrow \phi(R,\theta) = 0$,

$-\pi < \theta < \pi$.

- So we obtain:

(1.1) $\left.\begin{array}{ll} \Delta_2\phi + \lambda\phi = 0, & 0 < r < R \\ & -\pi < \theta < \pi \\ \phi(R,\theta) = 0 & \end{array}\right\}$ (1.2) $\left.\begin{array}{l} h''(z) = \lambda h(z) \\ h'(0) = 0 \end{array}\right\}$

Notice that (1.2) is <u>not</u> an eigenvalue problem, because it only has one BC.

However, (1.1) is the Helmholtz equation in a circle with Dirichlet BCs,

(1.1) $\left.\begin{array}{ll} \Delta_2\phi + \lambda\phi = 0 & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega \end{array}\right.$, with $\Omega \equiv$ circle

which we have solved before using Bessel's functions:

→ Eigenvalues: $\lambda_{n,m} = \left(\frac{z_{n,m}}{R}\right)^2$, with $z_{n,m}$ mth root of J_n.
$(n \geq 0,\ m \geq 1)$

→ Eigenfunctions: $\phi_{n,m}(r,\theta) = F_{n,m}(r)\,G_{n,m}(\theta)$, $G_{n,m}(\theta) = A_{n,m}\cos(n\theta) + B_{n,m}\sin(n\theta)$,

$F_{n,m}(r) = J_n\left(z_{n,m}\frac{r}{R}\right)$.

Remember how we solved (1.1): $\phi(r,\theta) = F(r)\,G(\theta) \rightarrow$

$$\frac{G(\theta)}{r}\frac{d}{dr}\left(r\frac{dF}{dr}\right) + \frac{F(r)}{r^2}\frac{d^2G}{d\theta^2} + \lambda F(r)G(\theta) = 0 \rightarrow$$

$$\rightarrow \frac{r}{F(r)}\frac{d}{dr}\left(rF'(r)\right) + \lambda r^2 = -\frac{G''(\theta)}{G(\theta)} = \mu \rightarrow$$

(1.1.1) $G''(\theta) = -\mu G(\theta)$

+ periodic B.C.

(1.1.2)

$$r^2F''(r) + rF'(r) + (\lambda r^2 - \mu)F(r) = 0$$

$$F(R) = 0$$

$$|F(0)| < +\infty .$$

(1.1.1) $\rightarrow \mu = n^2,\ n \geq 0$

$$G(\theta) = c_1\cos(n\theta) + c_2\sin(n\theta).$$

For $\mu = n^2$, (1.1.2) becomes the Bessel equation of order n, whose solutions are

[we first use the change of variables $z = \sqrt{\lambda}\,r$]

$$F(r) = c_1 J_n(\sqrt{\lambda}\,r) + c_2 Y_n(\sqrt{\lambda}\,r)$$

↳ we know $\lambda > 0$ using 2d Rayleigh quotient directly in (1.1).

↳ $|F(0)| < \infty \rightarrow c_2 = 0.$

↳ $F(R) = 0 = c_1 J_n(\sqrt{\lambda}\,r)$

defines the eigenvalues $J_n(\sqrt{\lambda}\,r) = 0 \leftrightarrow \sqrt{\lambda_{n,m}}\,r = z_{n,m}$ //

• Remark: To obtain the solution of (1.1), we have only used two facts about Bessel's functions:

→ its behaviour at zero,

→ that they have infinitely many roots.

Both things are summarised in their graphs $\left(\begin{matrix} \text{pages -172-, -173-} \\ \text{lect. 15 notes} \end{matrix} \right)$.

• With $\lambda_{n,m} = \left(\frac{z_{n,m}}{R} \right)^2 > 0$, we can now solve (1.2):

$$\left. \begin{array}{l} h_{n,m}(z) = C_{n,m} \cosh\left(\sqrt{\lambda_{n,m}}\, z\right) + D_{n,m} \sinh\left(\sqrt{\lambda_{n,m}}\, z\right) \\ h_{n,m}'(z) = 0 = D_{n,m} \sqrt{\lambda_{n,m}} \end{array} \right| \rightarrow$$

$$\rightarrow h_{n,m}(z) = \quad \cosh\left(\sqrt{\lambda_{n,m}}\, z\right).$$

• Then, superposition:

$$u_1(r,\theta,z) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \Phi_{n,m}(r,\theta)\, h_{n,m}(z) =$$

$$= \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} J_n\left(\sqrt{\lambda_{n,m}}\, r\right)\left(A_{n,m}\cos(n\theta) + B_{n,m}\sin(n\theta)\right)\cosh\left(\sqrt{\lambda_{n,m}}\, z\right).$$

To determine the coeff. $A_{n,m}$, $B_{n,m}$, we need to use the nonhomogeneous BC:

$$u_1(r,\theta,H) = \alpha(r,\theta) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \phi_{n,m}(r,\theta)\, h_{n,m}(H).$$

Remember that $\phi_{n,m}$ are the solutions to (1.1), thus they form a family of orthogonal eigenfunctions (in 2d).

↳ More exactly, $\phi^c_{n,m}(r,\theta) = J_n(\sqrt{\lambda_{n,m}}\, r)\cos(n\theta)$,

$$\phi^s_{n,m}(r,\theta) = J_n(\sqrt{\lambda_{n,m}}\, r)\sin(n\theta).$$

Therefore,

$$A_{n,m} = \frac{\int_\Omega \alpha\, \phi^c_{n,m}\, dA}{h_{n,m}(H) \int_\Omega (\phi^c_{n,m})^2\, dA},$$

$\Omega \equiv$ circle centered at the origin with radius R.

$$B_{n,m} = \frac{\int_\Omega \alpha\, \phi^s_{n,m}\, dA}{h_{n,m}(H) \int_\Omega (\phi^s_{n,m})^2\, dA},$$

• Solving ②: Laplace eq. in cylinder with

- zero temperature on top and lateral sides,
- given flux on bottom.

Repeating the steps from pages -181- on, we find that

$u_2(r,\theta,z) = \phi(r,\theta)\, h(z)$ (using same notation ϕ, h)

$$\text{(2.1)} \left.\begin{array}{ll} \Delta_2\phi + \lambda\phi = 0 & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega \end{array}\right\} \qquad \text{(2.2)} \left.\begin{array}{l} h''(z) = \lambda h(z) \\ h(H) = 0 \end{array}\right\}$$

(2.1) is exactly the same as (1.1) before $\Rightarrow \lambda > 0$ (page -182-).

$\hookrightarrow$ (2.2) $h_{n,m}(z) = \sinh\left(\sqrt{\lambda_{n,m}}\,(H-z)\right)$

Thus,

$$u_2(r,\theta,z) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} h_{n,m}(z)\left(A_{n,m}\,\phi^c_{n,m}(r,\theta) + B_{n,m}\,\phi^s_{n,m}(r,\theta)\right),$$

and

$$-\frac{\partial u_2}{\partial z}(r,\theta,0) = \beta(r,\theta) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} h'_{n,m}(0)\left(A_{n,m}\,\phi^c_{n,m}(r,\theta) + B_{n,m}\,\phi^s_{n,m}(r,\theta)\right)$$

$\Rightarrow A_{n,m},\ B_{n,m}$ //.

• Solving ③ : Laplace eq. in cylinder with

- zero temp. on top,
- insulation on bottom,
- given temperature on lateral sides.

→ This problem is different than the previous two ones: the nonhomogeneous BC is given on the lateral surface

↳ So the z problem has two homogeneous BCs, and thus it is an eigenvalue problem.

• We solve it from start:

$$u_3(r,\theta,z) = \phi(r,\theta)\, h(z) \rightarrow -\frac{\Delta_2 \phi}{\phi} = \frac{h''(z)}{h(z)} = \lambda$$

BCs:
$$\left.\begin{aligned} & u_3(r,\theta,H) = 0 = \phi(r,\theta)\, h(H) \rightarrow h(H) = 0 \\ & -\frac{\partial u_3}{\partial z}(r,\theta,0) = 0 = -\phi(r,\theta)\, h'(0) \rightarrow h'(0) = 0 \end{aligned}\right\}$$

(3.1) $\Delta\phi_2 + \lambda\phi = 0$, $0 < r < R$, $-\pi < \theta < \pi$

(+ periodicity + boundedness at $r = 0$)

↑ needs one more to be eigenvalue problem.

(3.2) $h''(z) = \lambda h(z)$

$h(H) = h'(0) = 0$

↑ eig. problem → determines λ.

(3.2) To avoid casing $\lambda<0, \lambda=0, \lambda>0, \dots$, we use the Rayleigh quotient to find that $\lambda<0$:

$$\lambda = \frac{\left. h h' \right]_0^H - \int_0^H (h')^2}{\int_0^H (h(z))^2 \, dz} \leq 0 .$$

If $\lambda = 0 \rightarrow h'(z) = 0 \rightarrow h(z) = c \rightarrow h(z) \equiv 0$ ↯.
$h(H) = 0$

Thus,

$$h(z) = c_1 \cos(\sqrt{-\lambda}\, z) + c_2 \sin(\sqrt{-\lambda}\, z)$$

$$h'(0) = c_2 \sqrt{-\lambda} = 0 \rightarrow c_2 = 0$$

$$h(H) = c_1 \cos(\sqrt{-\lambda}\, H) = 0 \rightarrow \lambda = -\left((2n-1)\frac{\pi}{2H}\right)^2, \; n \geq 1 .$$

- Summary (3.2):

$$\lambda_n = -\left(\frac{2n-1}{2}\frac{\pi}{H}\right)^2, \quad h_n(z) = \cos\left(\frac{2n-1}{2}\frac{\pi z}{H}\right), \; n \geq 1 .$$

- We proceed to solve (3.1). We separate variables

$$\left.\begin{array}{l} \phi(r,\theta) = F(r)G(\theta) \\ \Delta\phi = -\lambda_n \phi \end{array}\right\} \rightarrow \frac{r}{F(r)}\frac{d}{dr}\left(rF'(r)\right) + \lambda_n r^2 = -\frac{G''(\theta)}{G(\theta)} = \mu$$

(λ_n given, fixed)

3.1.1 $G''(\theta) = -\mu G(\theta)$ + periodic B.C.

3.1.2 $r^2 F''(r) + r F'(r) + (\lambda_n r^2 - \mu) F(r) = 0$, $|F(0)| < \infty$

↳ $\mu_m = m^2 \geq 0$

$$G(\theta) = c_1 \cos(m\theta) + c_2 \sin(m\theta).$$

• Finally, with $\mu_m = m^2$, $\lambda_n = -\left(\frac{2n-1}{2}\frac{\pi}{H}\right)^2$, we solve 3.1.2:

$$\left[\begin{array}{l} r^2 F''(r) + r F'(r) - \left(\underbrace{\left(\frac{(2n-1)\pi}{2H}\right)^2}_{|\lambda_n|} r^2 + m^2\right) F(r) = 0 \\ |F(0)| < \infty \end{array}\right. \quad [R]$$

↳ To get rid of the λ_n dependence, we change variables:

$\omega = \sqrt{|\lambda_n|}\, r$ [$|\lambda_n| \neq 0$ ⇝ if it was 0, solve that case separately: Euler-Cauchy ODE].

↓

The ODE becomes:

$$\omega^2 F''(\omega) + \omega F'(\omega) - (\omega^2 + m^2) F(\omega) = 0 \quad [M]$$

MODIFIED BESSEL ODE of order m.

↑ signs!

• This is again a second-order linear homogeneous ODE, and therefore the solution is any linear combination of two independent solutions.

Claim: The general solution of [M] is

$$F(w) = c_1 K_m(w) + c_2 I_m(w) \text{, where}$$

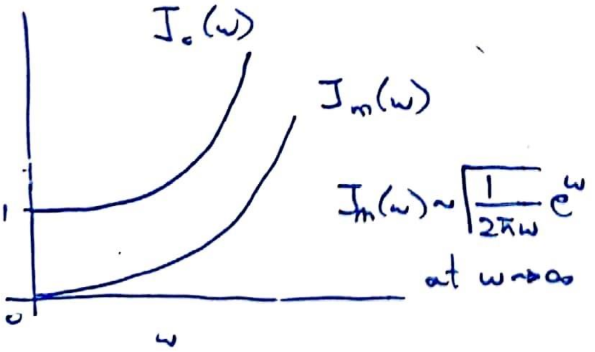
$I_m(w) \equiv$ modified Bessel function of order m of the first kind.

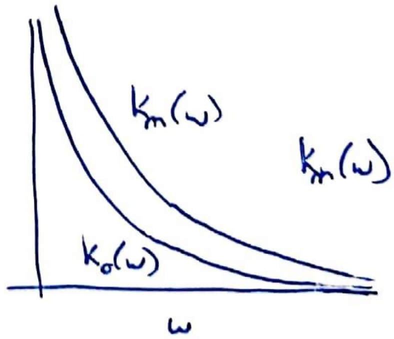
$K_m(w) \equiv$ " " " " " " " " second " .

We only need to know the following facts:

1) $I_m(w)$ is finite at $w=0$ but grows exponentially as $w \to \infty$.

2) $K_m(w)$ is singular at $w=0$ but decays exponentially to zero as $w \to \infty$.





$K_m(w) \sim \sqrt{\frac{\pi}{2}} \frac{e^{-w}}{\sqrt{w}}$ at $w \to \infty$.

$I_m(w) \sim \frac{1}{m!}\left(\frac{w}{2}\right)^m$ at $w \approx 0$

$$K_m(w) \sim \begin{cases} \ln(w) &, m = 0 \\ \frac{(m-1)!}{2}\left(\frac{2}{w}\right)^m &, m \neq 0 \end{cases} \quad \text{at } w \approx 0.$$

Remark: Things to remember: after separating variables, we will many times encounter linear ODEs which we don't know how to solve.

↳ But many are "well-known" and easy to find in google, wikipedia, books ... (Bessel, modified Bessel, Legendre, ...).

↳ The details and properties about these functions are interesting and can be "fun" exercises, but it is not worth at all trying to remember them.

↳ The "method of Frobenious" can be used to find these solutions in the form of series.

[it is a quite simple method, but kind of cumbersome].

[we are not covering it, but you are well-prepared to look it up if interested].

- Going back to page -189-, the solution to [R] is

$$F(r) = c_1 J_m(\sqrt{|\lambda_n|}\, r) + c_2 K_m(\sqrt{|\lambda_n|}\, r)$$

$$|F(0)| < \infty \to c_2 = 0$$

$$\to F(r) = c_1 J_m(\sqrt{|\lambda_n|}\, r)$$

- The solution to (3.1) is given by

$$G(\theta) \to \cos(m\theta), \sin(m\theta)$$

$$F(r) \to J_m(\sqrt{|\lambda_n|}\, r).$$

- Superposition:

$$u_3(r,\theta;z) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} J_m(\sqrt{|\lambda_n|}\, r)\left(A_{n,m}\cos(m\theta) + B_{n,m}\sin(m\theta)\right)h_n(z) =$$

$$= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\left(A_{n,m}\phi^c_{n,m}(r,\theta) + B_{n,m}\phi^s_{n,m}(r,\theta)\right)h_n(z),$$

with $\phi^c_{n,m}(r,\theta) = J_m(\sqrt{|\lambda_n|}\, r)\cos(m\theta)$,

$\phi^s_{n,m}(r,\theta) = J_m(\sqrt{|\lambda_n|}\, r)\sin(m\theta)$, $\sqrt{|\lambda_n|} = \frac{2n-1}{2}\frac{\pi}{H}$.

$h_n(z) = \cos(\sqrt{|\lambda_n|}\, z)$,

• The coeff. are determined from the nonhomogeneous BC,

$$u_3(R,\theta,z) = g(\theta,z) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} J_m(\sqrt{|\lambda_n|}R)\left(A_{n,m}\cos(m\theta)+B_{n,m}\sin(m\theta)\right)\cos(\sqrt{|\lambda_n|}z),$$

using orthogonality of sines and cosines.
(the coeff. with $m=0$ done apart as usual).

→ Finally, remember that the solution to [L] is the sum of u_1, u_2, u_3 //.