MATH 425

LECTURE 16

: Laplace Equation.

In this course we have focused on two PDEs: heat and wave equation (the standard PDEs of "parabolic" and "hyperbolic" type (*)). The stationary version of these PDEs, i.e., with $\partial_t \equiv 0$, is the Laplace equation:

$\Delta u = 0$, where $\Delta = \nabla \cdot \nabla$ is the Laplace operator,

or the inhomogeneous version of it, called the Poisson equation,

$\Delta u = f$.

- The easiest way to think of Laplace equation is as a PDE describing the equilibrium temperature distributions to the heat equation, but it also appears in probability (through Brownian motion) or complex analysis (in Cauchy-Riemann equations).

- The solutions to the Laplace equation are called harmonic functions.

(*) Laplace eq. is the standard "elliptic" PDE. The names parabolic, hyperbolic and elliptic PDEs comes from a nice, though quite useless, analogy with conics, in the case of second-order, linear, constant coeff. PDEs. [see Sect. 1.6]
The names are also used in a more broad context (including nonlinear PDEs), to associate PDEs with one of the three standard ones (heat, wave, Laplace) depending on the properties satisfied by their solutions.
(for ex., heat eq. solutions get smoother with time, they obey maximum principles, while wave equation solutions are of oscillatory character, have finite speed of propagation, ...).

- In this course, we will only study a few things about Laplace equation:

1) Solutions in rectangular geometry (2d, 3d) → Separation of variables exercise.

2) Solution in a circle → Separation of variables.

↳ Consequences: some important properties of harmonic functions.

- For further study, a natural continuation is Chapter 7, dealing with Green's functions, not covered in this course.

We will study instead the Fourier Transform and give an introduction to the topic of Calculus of Variations.

1. Laplace Equation in rectangles and cubes.

This first part is just an application of the method of separation of variables. It will help us in remembering it and as an introduction to higher dimensions.

Exercise 1: (2d)

PDE: $\Delta u = 0, \quad 0 < x < L,$
$\qquad\qquad\qquad\quad 0 < y < H,$

BC: $u(x,H) = g(x), \quad u_y(x,0) + u(x,0) = 0, \quad 0 < x < L,$
$\qquad u(0,y) = 0 = u_x(L,y), \quad 0 < y < H,$

Notice that here there isn't "initial condition" (there is no time variable), but we have one nonhomogeneous BC. This will play the role of "IC".

Separate variables: $u(x,y) = X(x)Y(y) \rightarrow$

$$X''(x)Y(y) + X(x)Y''(y) = 0 \rightarrow \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda \rightarrow$$

As usual, we obtain two ODEs,

① $X''(x) = -\lambda X(x)$, ② $Y''(y) = \lambda Y(y)$.

The BCs become

$$\left.\begin{aligned} 0 = u(0,y) &= X(0)Y(y) \text{ for } 0<y<H \to X(0)=0 \\ 0 = u_x(L,y) &= X'(L)Y(y) \quad \text{''} \quad \text{''} \quad \to X'(L)=0 \end{aligned}\right\}$$

$$0 = u_y(x,0) + u(x,0) = X(x)Y'(0) + X(x)Y(0) \to Y'(0)+Y(0)=0 ,$$

and we leave the nonhomogeneous one for the end (we cannot "separate" it).

Thus ① is an eigenvalue problem:

$$① \left.\begin{aligned} X''(x) &= -\lambda X(x) \\ X(0) &= X'(L) = 0 \end{aligned}\right\} \text{, whose solution we know:}$$

$$\lambda_n = \left(\frac{\pi}{L}\right)^2 \left(\frac{2n-1}{2}\right)^2 , \quad n \geq 1 ,$$

$$X_n(x) = \sin\left(\frac{2n-1}{2} \frac{\pi x}{L}\right) .$$

• With these values of λ_n, we solve ②:

$$② \left.\begin{aligned} Y''(y) &= \lambda_n Y(y) \\ Y'(0) + Y(0) &= 0 \end{aligned}\right\} \xrightarrow{\lambda_n > 0} Y_n(y) = A_n \cosh(\sqrt{\lambda_n}\, y) + B_n \sinh(\sqrt{\lambda_n}\, y)$$

$$Y_n'(y) = A_n \sqrt{\lambda_n} \sinh(\sqrt{\lambda_n}\, y) + B_n \sqrt{\lambda_n} \cosh(\sqrt{\lambda_n}\, y)$$

$Y_n'(0) + Y(0) = B_n\sqrt{\lambda_n} + A_n = 0 \Rightarrow A_n = -B_n\sqrt{\lambda_n}$.

Thus,

$$Y_n(y) = B_n\left(-\sqrt{\lambda_n}\cosh(\sqrt{\lambda_n}\, y) + \sinh(\sqrt{\lambda_n}\, y)\right)$$

- We apply the superposition principle:

$$u(x,y) = \sum_{n=1}^{\infty} B_n\left(-\sqrt{\lambda_n}\cosh(\sqrt{\lambda_n}\, y) + \sinh(\sqrt{\lambda_n}\, y)\right)\sin(\sqrt{\lambda_n}\, x),$$

where $\lambda_n = \left(\frac{\pi}{L}\frac{2n-1}{2}\right)^2$.

We find B_n using the nonhomogeneous BC:

$$u(x,H) = g(x) = \sum_{n=1}^{\infty} B_n\left(-\sqrt{\lambda_n}\cosh(\sqrt{\lambda_n}\, H) + \sinh(\sqrt{\lambda_n}\, H)\right)\sin(\sqrt{\lambda_n}\, x) \Rightarrow$$

$$\Rightarrow B_n = \frac{2}{L\left(-\sqrt{\lambda_n}\cosh(\sqrt{\lambda_n}\, H) + \sinh(\sqrt{\lambda_n}\, H)\right)}\int_0^L g(x)\sin(\sqrt{\lambda_n}\, x)\,dx.$$

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- Remark: In the above procedure we needed that only one BC was nonhomogeneous.

↳ But using the linearity property it is easy to deal with more:

<u>Example:</u> $$\left.\begin{aligned} &\Delta u = 0, \quad 0<x<L,\ 0<y<H, \\ &u(x,H) = g(x),\ u(x,0)=0, \\ &u(L,y)=0,\ u(0,y) = f(x) \end{aligned}\right\}$$

Let $u = u_1 + u_2$ with

$$\left.\begin{aligned} &\Delta u_1 = 0 \\ &u_1(x,H) = g(x),\ u_1(x,0)=0 \\ &u_1(L,y) = 0,\ u_1(0,y) = \underline{\underline{0}} \end{aligned}\right\} \quad \left.\begin{aligned} &\Delta u_2 = 0 \\ &u_2(x,H) = \underline{\underline{0}} = u_2(x,0), \\ &u_2(L,y) = 0,\ u_2(0,y) = f(x) \end{aligned}\right\}$$

You can check that u_1+u_2 solves the original problem.

[split in 3, 4 problems if you had 3, 4 nonhomog. BCs].

<u>Exercise 2:</u> (3d)

PDE: $\Delta u = 0, \quad 0<x<L,\ 0<y<H,\ 0<z<W,$

BC: $u(\pi, y, z) = g(y,z),$

$u(0,y,z) = 0,$

$u(x,0,z) = u(x,H,z) = 0,$

$u(x,y,0) = u(x,y,W) = 0$

(any other BC similarly)

[If more than one nonhomogeneous BC → split the problem]

<u>Solution:</u>

We have two apply separation of variables twice. Let's try with

$u(x,y,z) = X(x)Y(y)Z(z)$.

Then,

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0.$$

We need two separation constants, and we will obtain two eigenvalue problems. The BCs are

$$\left.\begin{array}{l} X(0) = 0, \\ Y(0) = Y(H) = 0, \\ Z(0) = Z(W) = 0 \end{array}\right\} \to \text{These give the BVP.}$$

- $\frac{Y''}{Y} = -\frac{X''}{X} - \frac{Z''}{Z} = -\lambda \Rightarrow$

① $\left.\begin{array}{l} Y'' = -\lambda Y \\ Y(0) = Y(H) = 0 \end{array}\right\}$ ② $\frac{Z''}{Z} + \frac{X''}{X} = \lambda \Rightarrow$

$$\Rightarrow \frac{Z''}{Z} = \lambda - \frac{X''}{X} = -\mu \Rightarrow$$

② $\left.\begin{array}{l} Z'' = -\mu Z \\ Z(0) = Z(W) = 0 \end{array}\right\}$ ③ $\left.\begin{array}{l} X'' = (\lambda+\mu) X \\ X(0) = 0 \end{array}\right\}$

① and ② are eigenvalue problems that determine λ and μ. Then, we can solve ③ for those values.

- We already know that

① $\rightarrow$ $\lambda_n = \left(\frac{n\pi}{H}\right)^2$, $n \geq 1$, $Y_n(y) = \sin\left(\frac{n\pi y}{H}\right)$,

② $\rightarrow$ $\mu_m = \left(\frac{m\pi}{W}\right)^2$, $m \geq 1$, $Z_m(z) = \sin\left(\frac{m\pi z}{W}\right)$.

and thus,

$$X_{n,m}(x) = A_{n,m}\cosh\left(\sqrt{\lambda_n+\mu_m}\,x\right) + B_{n,m}\sinh\left(\sqrt{\lambda_n+\mu_m}\,x\right), \;\; \rightarrow$$

$$X_{n,m}(0) = A_{n,m} = 0$$

$$\Rightarrow X_{n,m}(x) = B_{n,m}\sinh\left(\sqrt{\lambda_n+\mu_m}\,x\right).$$

- Superposition:

$$u(x,y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{n,m}\sinh\left(\sqrt{\left(\frac{n\pi}{H}\right)^2+\left(\frac{m\pi}{W}\right)^2}\,x\right)\sin\left(\frac{n\pi y}{H}\right)\sin\left(\frac{m\pi z}{W}\right),$$

with

$$u(\pi,y,z) = g(y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{n,m}\sinh\left(\sqrt{\lambda_n+\mu_m}\,\pi\right)\sin\left(\sqrt{\lambda_n}\,y\right)\sin\left(\sqrt{\mu_m}\,z\right),$$

which implies that [exercise: orthogonality]

$$B_{n,m} = \frac{4}{H\cdot W\cdot\sinh\left(\pi\sqrt{\lambda_n+\mu_m}\right)}\int_0^H\int_0^W g(y,z)\sin\left(\sqrt{\lambda_n}\,y\right)\sin\left(\sqrt{\mu_m}\,z\right)dy\,dz.$$

Remark: We can likewise solve the heat or wave eq. on rectangles (with two separation constants) or cubes (with three).

↳ same ideas, more calculus (... Math 241).

↳ For special geometries (like circles, cylinders, spheres,...) separation of variables is also explicit: the calculus gets much more involved (Bessel, modified Bessel, Legendre functions). [See Chapter 10]. (*)

↳ For general geometries, one needs more general theory and numerics (the eigenfunctions will not be explicit). [See Chapter 11 if interested]. (*)

• We will now solve Laplace equation in a circle, but only because the formula we will obtain is very useful to prove certain properties of harmonic functions.

(*) Mostly calculus (partially covered in M241). Useful in physics.

(*) Important generalization of what we have seen. Mostly for math majors.

Exercise 3: Poisson Formula.

PDE: $\Delta u = 0$ in Ω
BC: $u = h$ on $\partial\Omega$
Ω a circle of radius L.

Solution:

Polar coordinates: $\begin{cases} \Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0, & 0<r<L, \; -\pi<\theta<\pi, \\ u(L,\theta) = h(\theta) \end{cases}$

Separate variables: $u(r,\theta) = R(r)\varphi(\theta)$
Then,

$$R''\varphi + \frac{1}{r} R'\varphi + \frac{1}{r^2} R\varphi'' = 0 \Rightarrow r^2 \frac{R''}{R} + r\frac{R'}{R} + \frac{\varphi''}{\varphi} = 0 \Rightarrow$$

① $\varphi'' = -\lambda\varphi$ ② $r^2 R'' + rR' - \lambda R = 0$

What are the BC?

→ Periodicity in θ: $\varphi(\theta) = \varphi(\theta + 2\pi)$

This leads to

$$\varphi_n(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta), \quad n \geq 0, \quad \lambda_n = n^2.$$

Then, we solve ②:

$\left(r^0 \frac{d^0}{dr^0}\right)$

② $r^2 R_n'' + r R_n' - n^2 R = 0 \quad , \quad 0 < r < L.$

This is a linear, second-order ODE, and therefore its solution space is spanned by two independent solutions.

This ODE is called Euler-Cauchy ODE, with solution given by:

<u>$n > 0$:</u> Solutions of the form $r^p \rightarrow$

$$r^2 p(p-1) r^{p-2} + r p r^{p-1} - n^2 r^p = 0 \rightarrow p^2 = n^2 \rightarrow p = \pm n \rightarrow$$

$$\rightarrow R_n(r) = C_n r^n + D_n \frac{1}{r^n}.$$

<u>$n = 0$:</u> $R_0(r) = C_0 + D_0 \log(r)$.

→ We are interested in solutions that are bounded, so we must choose $D_n = D_0 = 0$ for all n.

↳ i.e., we are adding the BC. $|u(0,\theta)| < \infty$.

• Finally, superposition:

(renaming the constants).

$$[5] \quad u(r,\theta) = \sum_{n=0}^{\infty} r^n \left(A_n \cos(n\theta) + B_n \sin(n\theta)\right),$$

with

$$u(L,\theta) = \sum_{n=0}^{\infty} L^n \left(A_n \cos(n\theta) + B_n \sin(n\theta)\right),$$ which implies

that

[c]

$$A_n = \frac{1}{\pi L^n} \int_0^{2\pi} h(\theta)\cos(n\theta)\,d\theta \quad , \quad A_0 = \frac{1}{2\pi} \int_0^{2\pi} h(\theta)\,d\theta .$$

$$B_n = \frac{1}{\pi L^n} \int_0^{2\pi} h(\theta)\sin(n\theta)\,d\theta ,$$

- The interesting point of this problem is that we can rewrite the solution in closed form: we can summed the series explicitly (as we did with the Dirichlet Kernel).

First, substitute the coefficients [c] into the series [S]:

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} h(\phi)\,d\phi + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{r^n}{L^n}\int_0^{2\pi} h(\phi)\left(\cos(n\phi)\cos(n\theta) + \sin(n\phi)\sin(n\theta)\right)d\phi =$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{L}\right)^n \cos(n(\theta-\phi))\right)h(\phi)\,d\phi =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} P_L(r,\theta-\phi)\,h(\phi),$$ where

$$P_L(r,\theta) = 1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{L}\right)^n \cos(n\theta)$$ is the Poisson Kernel

That is, $u(r,\theta) = \frac{1}{2\pi}(P_L * h)(r,\theta)$.

By passing to complex variables and summing the geometric series that appear (exercise), one find that

$$P_L(r,\theta) = \frac{L^2-r^2}{L^2-r^2-2rL\cos(\theta)},$$

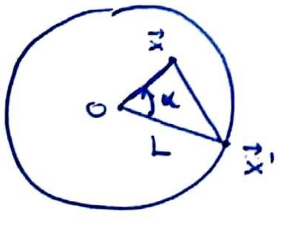
Poisson Kernel

and therefore,

$$u(r,\theta) = \frac{L^2-r^2}{2\pi}\int_0^{2\pi} \frac{h(\phi)}{L^2-r^2-2rh\cos(\theta-\phi)}\, d\phi$$

Poisson Formula.

- This formula has a nicer expression if we move back to Cartesian coordinates (almost valid for higher dimensions):

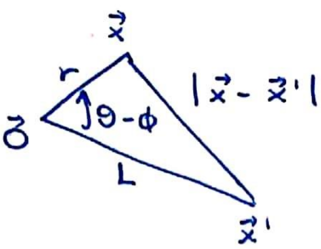


$\vec{x} \sim (r,\theta)$ fixed point

$\vec{x}' \sim (L,\phi)$ moving point

$\alpha = \theta - \phi$

Law of cosines,



$\hookrightarrow |\vec{x} - \vec{x}'|^2 = L^2 + r^2 - 2 L \, r \cos(\theta - \phi)$.

Thus,

$$\boxed{u(\vec{x}) = \frac{L^2 - |\vec{x}|^2}{2\pi} \int_{|\vec{x}'| = L} \frac{u(\vec{x}')}{|\vec{x} - \vec{x}'|^2} \frac{ds}{L}} , \qquad \text{POISSON'S FORMULA.}$$

where we $\vec{x}$ is a point in the circle $|\vec{x}|^2 = L^2$, the integral is a line integral ($ds = L \, d\phi$), and we have replaced h by $u(\vec{x}')$ to emphasize the meaning of Poisson formula:

Given the values of a harmonic function (i.e, one with Laplacian equal to zero) on a circle, Poisson formula gives the value of that function at any point inside the circle.

→ In next lecture we will use this formula to prove some properties of harmonic functions.