MATH 241 LECTURE 17: Higher Dimensional PDEs V (Spherical Harmonics)

In this lecture we focus on problems in spherical geometry, such as natural convection inside the earth (heat equation in a sphere) or seismic waves (wave eq.). Of course, the real processes are more complex, but the main mathematical component of most models are these "simple" linear PDEs.

↳ A good understanding of separation of variables is very useful when one works with cylindrical or spherical harmonics (modified Bessel functions, Legendre...) in numerical methods.

↳ for physics or engineering people, most times you simply use the code from someone else. A good understanding of what is expected to happen is necessary to check for error or to develop modifications/improvements.

- We are going to solve the wave equation with Dirichlet BCs, but basically the same applies for other problems (heat eq., Laplace eq....).

- **Problem:** Wave Equation in a sphere with Dirichlet BC.

PDE: $u_{tt} = c^2 \Delta u$ in Ω

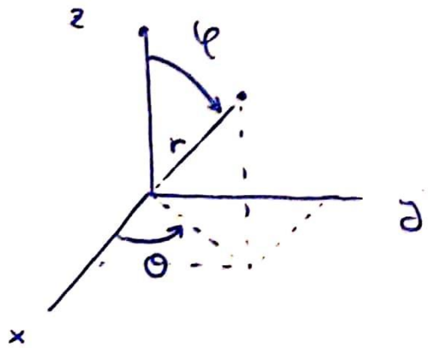
BC: $u = 0$ on $\partial\Omega$

I.C: $u|_{t=0} = \alpha$, $u_t|_{t=0} = \beta$

Ω sphere centered at origin with radius R.

Solution:

First, let's state the notation for the spherical coordinates: [the notation changes from books and websites, so always be careful].



$$\begin{cases} x = r \sin\varphi \cos\theta \\ y = r \sin\varphi \sin\theta \\ z = r\cos\varphi \end{cases} \qquad \begin{aligned} -\pi \le \theta \le \pi \\ 0 \le \varphi \le \pi \\ 0 \le r \le R \end{aligned}$$

math typical notation.

Remark: In physics they usually use φ, θ in the contrary way.

In many books the radial variable is denoted ρ.

In the book (R. Haberman), ϕ is used instead of φ

(I don't like it: it's less common and we have been using ϕ a lot to denote a 2d function of spatial variables).

- Remember that we can separate the time and spatial variables <u>without</u> knowing the geometry of Ω:

$u(r,\theta,\varphi,t) = \phi(r,\theta,\varphi)h(t) \rightsquigarrow$

$\rightsquigarrow \phi h'' = c^2 h \Delta\phi \rightsquigarrow \quad \frac{\Delta\phi}{\phi} = \frac{1}{c^2}\frac{h''}{h} = -\lambda \rightsquigarrow$

$$① \left.\begin{aligned} &\Delta\phi + \lambda\phi = 0 \quad \text{in } \Omega \\ &\phi = 0 \quad \text{on } \partial\Omega \end{aligned}\right\} \qquad ② \ h''(t) = -\lambda c^2 h(t)$$

where we have used that

$u(r,\theta,\varphi,t) = 0 = \phi(r,\theta,\varphi)h(t)$ for all $t \geq 0 \rightsquigarrow \phi \equiv 0$ on $\partial\Omega$.
and $(r,\theta,\varphi) \in \partial\Omega$

<u>Remark</u>: In ①, Δ denotes the 3-d Laplacian. The formulas for integration by parts are the same in

2d and 3d, so we can easily prove that $\lambda > 0$.

We do it here once:

$$\Delta\phi + \lambda\phi = 0 \rightarrow \int_{\Omega} \phi \Delta\phi \, dV = -\lambda \int_{\Omega} \phi^2 dV \rightarrow$$

$$\rightarrow \lambda = \frac{-1}{\int_{\Omega} \phi^2 dV} \left(- \int_{\Omega} |\nabla\phi|^2 dV + \underbrace{\int_{\partial\Omega} \cancel{\phi} \, \vec{n} \cdot \nabla\phi \, dS}_{BC.} \right) \geq 0.$$

If $\lambda = 0$, then $|\nabla\phi| = 0$ on $\Omega \rightarrow \phi = C \rightarrow \phi \equiv 0$ ⚡

$\phi = 0$ on $\partial\Omega$

→ In summary, the Helmholtz equation with Dirichlet BC. has strictly positive eigenvalues, both in 2d and 3d.

- Because $\lambda > 0$ (although unknown values yet),

$$h(t) = c_1 \cos(c\sqrt{\lambda}\, t) + c_2 \sin(c\sqrt{\lambda}\, t).$$

The values of λ will be given by solving the eigenvalue problem ①.

• Remark: 1) Problem ① will appear when solving the heat equation too.

2) For $\lambda \leq 0$, $\phi \equiv 0$ is the only solution.

3) If we had insulation BC, then $\lambda \geq 0$ and for $\lambda = 0$ we obtain $\phi \equiv c$.

4) If we have different BCs, the case $\lambda = 0$ might be interesting. It corresponds to solving the Laplace equation in a sphere
(which gives the equilibrium sol. to the heat eq.).

$\boxed{\text{Solving ①}}$ Helmholtz Equation in a sphere (Dirichlet BCs.).

$$\left.\begin{array}{ll} \Delta\phi + \lambda\phi = 0 & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega \end{array}\right\} \Omega \equiv \text{sphere}.$$

In spherical coordinates,

$$\Delta\phi(r,\theta,\varphi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial\phi}{\partial\varphi}\right) + \frac{1}{r^2\sin^2\varphi}\frac{\partial^2\phi}{\partial\theta^2}$$

• Now we try to separate variables:

$\phi(r,\theta,\varphi) = F(r)G(\theta)H(\varphi)$ and substitute in the PDE,

$$\overbrace{\frac{G(\theta)H(\varphi)}{r^2}\frac{d}{dr}\left(r^2F'(r)\right)+\frac{F(r)G(\theta)}{r^2\sin\varphi}\frac{d}{d\varphi}\left(\sin\varphi\, H'(\varphi)\right)+\frac{F(r)H(\varphi)}{r^2\sin^2\varphi}G''(\theta)}^{\Delta\phi(r,\theta,\varphi)}+$$

$$+\lambda F(r)G(\theta)H(\varphi)=0 \rightarrow \left\{\begin{array}{l}\text{multiply by } r^2\sin^2\varphi,\\ \text{divide by } FHG\end{array}\right\}$$

$$\frac{\sin^2\varphi}{F(r)}\frac{d}{dr}\left(r^2F'(r)\right)+\frac{\sin\varphi}{H(\varphi)}\frac{d}{d\varphi}\left(\sin\varphi\, H'(\varphi)\right)+\frac{G''(\theta)}{G(\theta)}+\lambda r^2\sin^2\varphi=0 \rightarrow$$

$\rightarrow$ (1.1) $\left.\begin{array}{l} G''(\theta)=-\eta G(\theta)\\ +\text{ periodic BC.}\end{array}\right\}$ ($\theta \equiv$ horizontal angle, $-\pi\le\theta\le\pi$)

(1.2) $$\frac{1}{F(r)}\frac{d}{dr}\left(r^2F'(r)\right)+\frac{1}{\sin\varphi\, H(\varphi)}\frac{d}{d\varphi}\left(\sin\varphi\, H'(\varphi)\right)+\lambda r^2=\frac{\eta}{\sin^2\varphi}$$

- The BC $\phi=0$ on $\partial\Omega$ translates to:

$$\phi(R,\theta,\varphi)=F(R)G(\theta)H(\varphi)=0 \text{ for all } \theta,\varphi \rightarrow F(R)=0.$$

<u>Remark</u>: (1) is a 3d problem, so we will need two separation constants (not counting λ).

• The solution to (1.1) is well-known:

$\eta_n = n^2$, $n \geq 0$, with $G(\theta) = c_1 \cos(n\theta) + c_2 \sin(n\theta)$.

• With $\eta_n = n^2$, we proceed to solve (1.2). We rewrite it trying to separate r and φ:

(1.2) $$\frac{1}{F(r)}\frac{d}{dr}\left(r^2 F'(r)\right) + \lambda r^2 = -\frac{1}{\sin\varphi\, H(\varphi)}\frac{d}{d\varphi}\left(\sin\varphi\, H'(\varphi)\right) + \frac{n^2}{\sin^2\varphi} = \mu$$

which gives rise to:

(1.2.1) $$\left.\frac{d}{d\varphi}\left(\sin\varphi\, H'(\varphi)\right) + \left(\mu \sin\varphi - \frac{n^2}{\sin\varphi}\right)H(\varphi) = 0\right\}$$

(1.2.2) $$\left.\begin{aligned}&\frac{d}{dr}\left(r^2 F'(r)\right) + \left(\lambda r^2 - \mu\right)F(r) = 0\\ &F(R) = 0\end{aligned}\right\}$$

• We choose to solve first (1.2.1) because it doesn't depend on λ. This problem will determine the values of μ. Then, with those values of μ, we will use (1.2.2) to find the values of λ.

→ It is clear than for (1.2.1) and (1.2.2) there must be some "hidden" boundary conditions.

- Solving (1.2.1) : Legendre Polynomials and Associated Legendre functions.

The linear second-order ODE (1.2.1) is of Sturm-Liouville type with $p(\varphi) = \sin\varphi$, $\sigma(\varphi) = \sin\varphi$, $q(\varphi) = -\frac{n^2}{\sin\varphi}$.

It is not a regular one due to the points $\varphi = 0$ and $\varphi = \pi$, which corresponds to the North and South poles. Notice that this is simply a singularity coming from the spherical coordinates (like $r=0$ in polar coord.).

To have meaningful solutions, we must ensure that the solutions are bounded at those points:

$|H(0)| < \infty$, $|H(\pi)| < \infty$

So (1.2.1) is the BVP given by

$$\begin{cases} \frac{d}{d\varphi}\left(\sin\varphi\, H'(\varphi)\right) + \left(\mu \sin\varphi - \frac{n^2}{\sin\varphi}\right) H(\varphi) = 0, & 0 < \varphi < \pi, \\ |H(0)| < \infty, \quad |H(\pi)| < \infty \end{cases}$$

This is a new ODE, which in most book appears in a different form:

Change of variables: $x = \cos\theta$ (bijection from $[0,\pi]$ to $[-1,1]$).

The equation becomes

$$\boxed{\frac{d}{dx}\left((1-x^2)H'(x)\right) + \left(\mu - \frac{n^2}{1-x^2}\right)H(x) = 0 \qquad -1 < x < 1}$$

Associated Legendre ODE. [L]

Claim: The general solution to [L] is given by

$H(x) = c_1 P^n_\alpha(x) + c_2 Q^n_\alpha$, where $\mu = \alpha(\alpha+1)$ and

$\underbrace{P^n_\alpha, Q^n_\alpha}$ are the associated Legendre functions of the first and second kind, respec.

More commonly known as Spherical Harmonics.

→ Both P^n_α, Q^n_α are generally unbounded at $x = 1$ or $x = -1$.

The only ones that are bounded both at $x = 1$ and $x = -1$ are the spherical harmonics of first kind with $\alpha = m$ an integer:

(Hard to prove).

$|P^n_m(1)| < \infty$, $|P^n_m(-1)| < \infty$.

- We focus on the case $\alpha = m$ since we are only interested in bounded solutions.

It is surprising that the spherical harmonics of first kind with integer $\alpha = m$ are quite simple functions: they are polynomials!

1) Legendre polynomials: $P_m^0(x)$

These satisfy the ODE $\left.\begin{array}{l}\frac{d}{dx}\left((1-x^2)H'(x)\right) + m(m+1)H(x) = 0 \\ \text{with } |H(1)|<\infty,\ |H(-1)|<\infty\end{array}\right\}$

They satisfy Rodrigues's formula:

$$\boxed{P_m^0(x) = \frac{1}{2^m n!}\frac{d^m}{dx^m}(x^2-1)^m} \Rightarrow P_m^0 \text{ is a polynomial of degree } m.$$

2) Associated Legendre polynomials: $P_m^n(x)$

$$\boxed{P_m^n(x) = (x^2-1)^{n/2}\frac{d^n P_m^0}{dx^n}}$$

- Remark: Since P_m^0 is a polynomial of degree m,

$P_m^n(x) \equiv 0$ for $n > m$.

• Summary : The solution to the eigenvalue problem

$$\begin{cases} \frac{d}{dx}\left((1-x^2)H'(x)\right) + \left(\mu - \frac{n^2}{1-x^2}\right)H(x) = 0, & -1 < x < 1, \\ |H(-1)| < \infty, \ |H(1)| < \infty. \end{cases}$$

is given by $\left| \begin{array}{l} \mu_{n,m} = m(m+1) \text{ with } m \geq n \\ H_{n,m}(x) = P_m^n(x) \end{array} \right.$

• Back to (1.2.1) (page -201-) : $H_{n,m}(\varphi) = P_m^n(\cos\varphi)$
$[x = \cos\varphi]$

That is, the solution to the eigenvalue problem (1.2.1) is

$$\left| \begin{array}{l} \mu_{n,m} = m(m+1) \text{ with } m \geq n, \\ H_{n,m}(\varphi) = P_m^n(\cos\varphi) \end{array} \right.$$

Remark: The orthogonality of these eigenfunctions is easy to prove from the Associated Legendre ODE. (good exercise).

• Solving 1.2.2: Spherical Bessel Functions.

Now we know that $\mu = m(m+1)$, so we have to solve

$$\left.\begin{aligned}&\frac{d}{dr}\left(r^2 F'(r)\right) + \left(\lambda r^2 - m(m+1)\right)F(r) = 0, \quad 0 < r < R\\ &F(R) = 0, \quad |F(0)| < \infty\end{aligned}\right\}$$

↑ added as usual

This ODE reminds Bessel's ODE, so let's first do the same change to get rid of the λ dependence:

$x = \sqrt{\lambda}\, r$ (we know that $\lambda > 0$, so this is a bijection).

$$\boxed{x^2 F''(x) + 2x F'(x) + \left(x^2 - m(m+1)\right)F(x) = 0}$$ Spherical Bessel ODE.

The general solution is

$F(x) = c_1 j_m(x) + c_2 y_m(x)$, where

$$j_m(x) = \frac{1}{\sqrt{x}} J_{m+\frac{1}{2}}(x)$$

$$y_n(x) = \frac{1}{\sqrt{x}} Y_{m+\frac{1}{2}}(x)$$

are called Spherical Bessel functions (of the first and second kind).

(*) Bessel functions of order $m+\frac{1}{2}$ look like the ones we have studied before.

$\sqcap$ It is just a matter of substitution to check that j_m, y_m solve the Spherical Bessel ODE, using that $J_{m+\frac{1}{2}}, Y_{m+\frac{1}{2}}$ solve Bessel ODE.

That is, if $J_{m+\frac{1}{2}}, Y_{m+\frac{1}{2}}$ are solutions of

$$x^2 F''(x) + x F'(x) + \left(x^2 - (m+\tfrac{1}{2})^2\right) F(x) = 0,$$

then j_m, y_m solve $x^2 F''(x) + 2xF'(x) + (x^2 - (m(m+1)) F(x) = 0$ $\sqcup$

Remark: The functions j_m, y_m behave and look very similar to J_m, Y_m ("same" graphs).

- Going back to r:

$$F(r) = c_1 j_m(\sqrt{\lambda}\, r) + c_2 y_m(\sqrt{\lambda}\, r)$$

$$|F(0)| < \infty \Rightarrow c_2 = 0$$

$$F(R) = 0 \Leftrightarrow j_m(\sqrt{\lambda} R) = 0 \Leftrightarrow J_{m+\frac{1}{2}}(\sqrt{\lambda} R) = 0 \Leftrightarrow$$

$$\Leftrightarrow \sqrt{\lambda_{m,\ell}}\, R = z_{m,\ell}\text{, with } z_{m,\ell} \text{ the } \ell^{th} \text{ root of } J_{m+\frac{1}{2}}.$$

• Remark: (not needed) A nice formula for the spherical Bessel functions is

$$j_m(x) = \frac{1}{\sqrt{x}} J_{m+\frac{1}{2}}(x) = x^m \left(\frac{-1}{x}\frac{d}{dx}\right)^m \left(\frac{\sin x}{x}\right).$$

In particular, $j_0(x)$ is the $\mathrm{Sinc}(x)$ function: $\frac{\sin x}{x}$.

• Joining all together:

(1.1) $\eta_n = n^2$, $G(\theta) = c_1 \cos(n\theta) + c_2 \sin(n\theta)$,

(1.2.1) $\mu_{n,m} = m(m+1)$ with $m \geq n$,

$H_{n,m}(\varphi) = P_m^n(\cos\varphi)$.

(1.2.2) $\lambda_{m,\ell} = \left(\frac{z_{m,\ell}}{R}\right)^2$, with $z_{m,\ell}$ the ℓ^{th} root of $J_{m+\frac{1}{2}}$,

$F(r) = \frac{1}{\sqrt{r}} J_{m+\frac{1}{2}}(r) \equiv j_m(r)$.

Superposition:

$$u(r,\theta,\varphi,t) = \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} \sum_{\ell=1}^{\infty} \left(A_{n,m,\ell} \cos\left(c\sqrt{\lambda_{m,\ell}}\, t\right) + B_{n,m,\ell} \sin\left(c\sqrt{\lambda_{m,\ell}}\, t\right)\right) \cdot$$

$$\cdot \left(C_{n,m,\ell} \cos(n\theta) + D_{n,m,\ell} \sin(n\theta)\right) j_m(r) P_m^n(\cos\varphi).$$

I.C.:

$$\left.\begin{aligned} u(r,\theta,\varphi,0) &= \alpha(r,\theta,\varphi) \\ u_t(r,\theta,\varphi,0) &= \beta(r,\theta,\varphi) \end{aligned}\right\} \rightsquigarrow A, B, C, D //$$