MATH 241

LECTURE 18 : Nonhomogeneous Problems I.

(~ Sect. 8.2)

So far in the course we have used one method to solve PDE: separation of variables.

Two conditions were needed: linear and homogeneous PDE.

We will study now how to adapt it to deal with nonhomogeneous problems.

1. Time-independent nonhomogeneous terms.

Let's start with a particular (simpler) case. Consider the heat equation with steady nonhomogeneous terms. That is,

Problem:

$$\left.\begin{array}{lll} \text{PDE} & u_t = k\,u_{xx} + Q(x), & 0 < x < L \\ \text{BC} & u(0,t) = A & \\ & u(L,t) = B & \\ \text{IC} & u(x,0) = f(x) & \end{array}\right\}$$

Remark: In higher dimensions A, B could be functions of the spatial variables, not just constants.

Solution:

The idea is to transform this problem into one that is homogeneous.

Because the force $Q(x)$ and the BCs A, B are time-independent, we expect that the problem has an equilibrium solution. Let's reformulate the problem around that equilibrium solution:

$$u_E(x) \equiv \text{solution of} \quad \left.\begin{aligned} &k\, v''(x) + Q(x) = 0, \quad 0<x<L, \\ &v(0) = A, \; v(L) = B \end{aligned}\right\}$$

Then, $w(x,t) = u(x,t) - u_E(x)$ satisfies the following problem:

$$\left.\begin{aligned} u = w + u_E \;\rightarrow\; & u_t = w + \frac{\partial}{\partial t} u_E \\ \rightarrow\; & k u_{xx} + Q(x) = k w_{xx} + \underbrace{k u_E'' + Q(x)}_{=0} \end{aligned}\right\} \rightarrow$$

$$\left\{ u_t = k u_{xx} + Q \right\} \rightarrow \quad w = k w_{xx},$$

with $w(0,t) = u(0,t) - u_E(0) = A - A = 0,$

$w(L,t) = u(L,t) - u_E(L) = B - B = 0;$

and $w(x,0) = u(x,0) - u_E(x) = f(x) - u_E(x).$

• That is, w solves
$$\left.\begin{aligned} w_t &= k w_{xx} \\ w(0) &= 0 = w(L) \\ w(x,0) &= f(x) - u_E(x) \end{aligned}\right\}$$

Now we can solve this problem using separation of variables as usual.

• <u>Summary</u>: To deal with <u>time-independent</u> BCs, steps:

1) Find the equilibrium solution (if it exists).
2) Write the homogeneous problem for $w = u - u_E$.
3) Solve by separation of variables.

• <u>Remark</u>: There might not be an equilibrium solution. In that case we cannot use this idea.

More difficult is the case when the nonhomogeneous terms depend on time.

2. Time-dependent nonhomogeneous terms.

We have two types of nonhomogeneities: the forcing term and the BCs. We are going to see now that we can always transform the problem into one with homogeneous BCs.

We show it for the heat equation:

$$\left.\begin{array}{ll} \text{PDE}: & u_t = K u_{xx} + Q(x,t), \quad 0<x<L \\ \text{BCs}: & u(0,t) = A(t) \\ & u(L,t) = B(t) \\ \text{IC}: & u(x,0) = f(x). \end{array}\right\} [P]$$

The idea is to use again linearity: split the solution into two parts,

$u(x,t) = v(x,t) + r(x,t)$,

with $r(x,t)$ any smooth function satisfying the BCs,

$r(0,t) = A(t)$, $r(L,t) = B(t)$.

Because the BCs are linear, it is clear that

$v(0,t) = u(0,t) - r(0,t) = 0$, $v(L,t) = \ldots = 0$.

Substituting $u = v + r$ into the PDE we obtain the problem for v:

$u_t = v_t + r_t$, $u_{xx} = v_{xx} + r_{xx}$ $\rightsquigarrow$

$$\left.\begin{aligned} v_t &= k v_{xx} + Q(x,t) + k r_{xx} - r_t \\ v(0,t) &= v(L,t) = 0 \\ v(x,0) &= u(x,0) - r(x,0) \end{aligned}\right\}$$

$\to$ $r(x,t)$ was any smooth function satisfying the original BCs, so we simply choose one and it is thus a known function.

For example, take it linear in x: $r(x,t) = A(t) + \frac{x}{L}\left(B(t) - A(t)\right)$.

[Of course, $r(x,t)$ must be chosen in each problem depending on the kind of BC.]

- In summary, we have transformed [P] into

$$\left.\begin{aligned} v_t &= k v_{xx} + \tilde{Q}(x,t) \\ v(0,t) &= v(L,t) = 0 \\ v(x,0) &= g(x) \end{aligned}\right\} \text{ with } \quad \begin{aligned} \tilde{Q} &= Q + k r_{xx} - r_t, \\ g(x) &= f(x) - r(x,0) \end{aligned}$$

On next lecture we will study the "method of eigenfunction expansion" to solve problem with forcing term and homogeneous BCs (since we can always reduce the nonhomogeneous BCs to forces).

Remark: These ideas work the same in higher dimensions.

→ But more difficult to find an equilibrium solution (we don't know yet how to solve the Laplace eq. with a force).

→ It can also be difficult to find a solution r satisfying the BCs.