MATH 241
LECTURE 19

: Nonhomogeneous Problems II

(~ Section 8.3)

In Lecture 18 we found a simple way to "move" the nonhomogeneous terms on the BCs to forcing terms. So we just need to worry now in how to solve problems with (time-dependent) forces.

- <u>Method of eigenfunction Expansion</u>

We first describe it for the heat equation (1d).

$$[P] \begin{cases} \text{PDE}: & u_t = k\, u_{xx} + Q(x,t), \quad 0 < x < L, \\ \text{BC}: & u(0,t) = u(L,t) = 0 \\ \text{IC}: & u(x,0) = f(x) \end{cases}$$

(← possibly after transforming the original problem; Lect. 18).

Here $Q(x,t)$ is a known function of x and t (<u>data</u>)

The idea is to use generalized Fourier Series to transform the PDE into a ODE only in time.

Review: Linear, first-order ODEs.

$$\left.\begin{array}{l} f'(t) + p f(t) = g(t) \\ f(0) = f_0 \end{array}\right\} \quad (p \text{ constant})$$

Integrating factor: $e^{pt} \rightsquigarrow \frac{d}{dt}\left(e^{pt} f(t)\right) = e^{pt}\left(f'(t) + p f(t)\right) \rightsquigarrow$

$$\rightsquigarrow \frac{d}{dt}\left(e^{pt} f(t)\right) = e^{pt} g(t) \rightsquigarrow$$

$$\rightsquigarrow e^{pt} f(t) - f(0) = \int_0^t e^{p\tau} g(\tau)\, d\tau \rightsquigarrow$$

$$\rightsquigarrow \boxed{f(t) = f_0 e^{-pt} + \int_0^t e^{-p(t-\tau)} g(\tau)\, d\tau}$$

- Main idea: "Fourier series transforms derivatives in space to simple multiples of the function".

↳ Roughly, if ϕ_n are eigenfunctions from $\left.\begin{array}{l}\phi'' = -\lambda\phi \\ \text{plus BC}\end{array}\right\}$,

and we write $f(x) = \sum_n c_n \phi_n(x)$, then,

$$\underbrace{f''(x)}_{} = \sum_n c_n \phi_n''(x) = - \sum_n c_n \lambda_n \phi_n(x).$$

derivatives in space "disappear".

Let's show the steps for problem [P].

① Write the associated homogeneous problem.

In this case,
$$\left.\begin{aligned} &u_t = k\, u_{xx}, \quad 0<x<L \\ &u(0,t) = u(L,t) = 0 \\ &u(x,0) = f(x) \end{aligned}\right\}$$

② Find the eigenfunctions that will appear when separating variables in ①.

In this case, they would be the solutions to

$$\left.\begin{aligned} &\phi''(x) = -\lambda \phi(x) \\ &\phi(0) = \phi(L) = 0 \end{aligned}\right\} \rightsquigarrow \underline{\lambda_n, \phi_n}$$

<u>Remark</u>: In general, this step might not yield explicit eigenfunctions, but hopefully we obtain a Sturm-Liouville problem.

↳ In any case, we will need the <u>completeness</u> and <u>orthogonality</u> properties, that is, that any (piecewise smooth) function can be written as an infinite expansion of the eigenfunctions, and that these eigenfunctions are orthogonal (so that we can compute the coefficients.

→ In real life problems, checking the completeness property can be very hard. Many times, one simply assumes it and check if the solutions are valid.

③ Write the generalized Fourier Series of $u(x,t)$ and $Q(x,t)$. More specifically, for each fixed time, think of $u(x,t)$ and $Q(x,t)$ as functions of the espatial variable x. Expand these functions in terms of the eigenfunctions $\phi_n(x)$:

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x),$$

$$Q(x,t) = \sum_{n=1}^{\infty} b_n(t)\phi_n(x).$$

Notice that t is treated as a parameter, and as such the Fourier coefficients depend on it.

Remark: We started the sum at $n=1$ because it corresponds here to the first eigenvalue. In general, the sum runs over all the eigenvalues.

- Since $Q(x,t)$ is given, we can compute the coefficients $b_n(t)$ as usual:

$$b_n(t) = \frac{\int_0^L Q(x,t)\phi_n(x)\,dx}{\int_0^L (\phi_n(x))^2\,dx}.$$

Remark: Stop a moment to check that you see that so far the method should work exactly the same for S-L problems, or even more for higher dimensional PDEs.

Remark: We can compute the coeff. $a_n(t)$ in terms of $u(x,t)$, but since $u(x,t)$ is our unknown, this does not seem very wise.

④ Write the problem [P] in terms of the Fourier coeff. $a_n(t)$. To do so, substitute u and Q by their Fourier Series and apply orthogonality.

Substitution: $u(x,t) = \sum_{n=1}^{\infty} a_n(t)\,\phi_n(x)$,

$u_t(x,t) = \sum_{n=1}^{\infty} a_n'(t)\,\phi_n(x)$, "found before, so data now"

[*]
(page -222-) $u_{xx}(x,t) = \sum_{n=1}^{\infty} a_n(t)\,\phi_n''(x) = -\sum_{n=1}^{\infty} \lambda_n a_n(t)\,\phi_n(x)$.

main point

↳ this will depend on the eigenvalue problem (point ②), but it will always get rid of the derivatives in space.

Therefore,

$$\sum_{n=1}^{\infty} a_n'(t)\,\phi_n(x) = -k\sum_{n=1}^{\infty} \lambda_n a_n(t)\,\phi_n(x) + \sum_{n=1}^{\infty} b_n(t)\,\phi_n(x).$$

Finally, we project this equation onto each eigenfunction. To do so, multiply by $\phi_m(x)$ and integrate, taking into account the orthogonality of eigenfunctions:

$$a_n'(t) + k\lambda_n a_n(t) = b_n(t)$$ [we have divided by $\int_0^L (\phi_n(x))^2 dx \neq 0$].

↳ linear first-order ODE for a_n.

Initial data: $u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n(0)\phi_n(x) \Rightarrow$

$$\Rightarrow a_n(0) = \frac{\int_0^L f(x)\phi_n(x)dx}{\int_0^L (\phi_n(x))^2 dx}.$$

Therefore, we can solve this ODE to find

$$[S] \quad a_n(t) = a_n(0)e^{-\lambda_n k t} + \int_0^t e^{-\lambda_n k(t-\tau)} b_n(\tau)d\tau.$$

Conclussion: The solution to [P] is

$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x)$ with $a_n(t)$ given by [S],

where $a(0) = \frac{\int_0^L f(x)\phi_n(x)dx}{\int_0^L (\phi_n(x))^2 dx}$, $b_n(t) = \frac{\int_0^L Q(x,t)\phi_n(x)dx}{\int_0^L (\phi_n(x))^2 dx}$

→ Remark: The BCs are automatically satisfied because they are included in the problem that defines ϕ_n !!

Remark: This method solves completely non-homogeneous (linear) problems, no matter if the nonhomogeneous terms are time independent or not.

It also works when there isn't an equilibrium solution.

Remark: What if we deal with the wave equation?

Check that everything works similarly, but you will obtain a second-order, linear ODE for $a_n(t)$.

(with I.C. $a_n(0)$, $a_n'(0)$).

[*] We will assume that term-by-term differentiation of these infinite series is valid.

↳ in general one should be careful, because it is not always a valid operation (topic for analysis course).

Remark: If $Q \equiv 0$ (i.e., $b_n(t) = 0$), then $a_n(t) = a_n(0) e^{-\lambda_n k t}$,

with $a_n(0)$ the Fourier coeff. of $f(x)$. That is, we recover the solution for the homogeneous problem.