MATH 241 : Practice Midterm, Solutions.

P1

Part a)

a.1) If Φ solves $\left.\begin{array}{ll}\Delta\Phi = f & \text{in } \Omega \\ \vec{n}\cdot\nabla\Phi = 0 & \text{on } \partial\Omega\end{array}\right\}$, then

$\int_\Omega \Delta\Phi \, dV = \int_\Omega f \, dV$. Using integration by parts, we obtain that

$$\int_\Omega f \, dV = \int_{\partial\Omega} \vec{n}\cdot\nabla\Phi \, dS = 0$$

a.2) The above problem corresponds to the equilibrium (time-independent) solution to

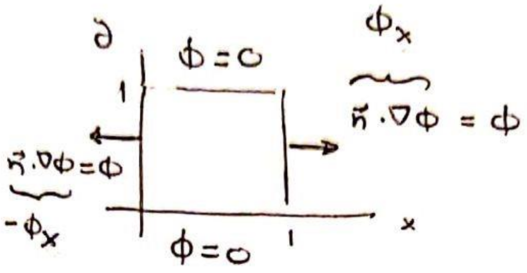
$$\left.\begin{array}{ll}\Phi_t - \Delta\Phi = -f & \text{in } \Omega \\ \vec{n}\cdot\nabla\Phi = 0 & \text{on } \partial\Omega\end{array}\right\}$$

which describes the evolution of the temperature in the region Ω. Since the boundary is insulated, for an equilibrium to exist the total heat generation in Ω

must be zero, i.e., $\int_{\Omega} f \, dV = 0$.

• Part b)

$$\left.\begin{array}{ll} \Delta\phi + \lambda\phi = 0 & \text{in } \Omega \\ \vec{n}\cdot\nabla\phi = \phi & \text{on } \partial\Omega_1 \\ \phi = 0 & \text{on } \partial\Omega_2 \end{array}\right\}$$



b.1)
$$\left.\begin{array}{l} \phi_{xx} + \phi_{yy} + \lambda\phi = 0 \quad \text{in } 0<x<1,\ 0<y<1, \\ \phi(x,0) = 0 = \phi(x,1), \\ -\phi_x(0,y) = \phi(0,y) \\ \phi_x(1,y) = \phi(1,y) \end{array}\right\} \quad [E]$$

b.2) The Rayleigh quotient doesn't say much:

$$\lambda = \frac{-\int_{\partial\Omega} \phi\, \vec{n}\cdot\nabla\phi + \int_{\Omega} |\nabla\phi|^2}{\int_{\Omega} \phi^2} = \frac{-\int_{\partial\Omega_1} \phi^2\, dl + \int_{\Omega} |\nabla\phi|^2\, dA}{\int_{\Omega} \phi^2\, dA} \quad ?$$

So we check it directly: $\lambda = 0$ is an eigenvalue

if $\left.\begin{array}{ll} \Delta\phi = 0 & \text{in } \Omega \\ \vec{n}\cdot\nabla\phi = \phi & \text{on } \partial\Omega_1 \\ \phi = 0 & \text{on } \partial\Omega_2 \end{array}\right\}$ has a solution $\phi \neq 0$.

We try to solve this problem [E] by separation of variables:

$$\phi(x,y) = F(x)G(y) \rightarrow \frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)} = \mu$$

BC:

$$\phi(x,0) = 0 = \phi(x,1) \Rightarrow G(0) = G(1) = 0$$

$$-\phi_x(0,y) = \phi(0,y) \Rightarrow -F'(0)G(y) = F(0)G(y) \Rightarrow$$

$$\Rightarrow -F'(0) = F(0),$$

$$\phi_x(1,y) = \phi(1,y) \rightarrow F'(1) = F(1).$$

Thus, ① $\left.\begin{array}{l} G''(y) + \mu G(y) = 0 \\ G(0) = G(1) = 0 \end{array}\right\}$ ② $\left.\begin{array}{l} F''(x) - \mu F(x) = 0 \\ -F'(0) = F(0) \\ F'(1) = F(1) \end{array}\right\}$

We know that the solution to ① is

$\mu_n = (n\pi)^2$, $n \geq 1$, with $G_n(y) = \sin(n\pi y)$.

For each $\mu_n = (n\pi)^2$, $n \geq 1$, we check if $F(x)$ has a solution:

$$F_n(x) = c_1 \cosh(n\pi x) + c_2 \sinh(n\pi x),$$

$$F_n'(x) = c_1 n\pi \sinh(n\pi x) + c_2 n\pi \cosh(n\pi x),$$

$$-F'(0) = F(0) \Leftrightarrow -c_2 n\pi = c_1$$

$$\left. \begin{aligned} F'(1) = F(1) \Leftrightarrow \; & c_1 n\pi \sinh(n\pi) + c_2 n\pi \cosh(n\pi) = \\ & = c_1 \cosh(n\pi) + c_2 \sinh(n\pi) \end{aligned} \right\} \Leftrightarrow$$

$$\Leftrightarrow -\cancel{c_2}(n\pi)^2 \sinh(n\pi) + \cancel{c_2} n\pi \cosh(n\pi) = \qquad \left\{ \begin{array}{c} \text{if } c_2 = 0, \text{ then} \\ c_1 = 0 \Rightarrow F = 0 \\ \Rightarrow \phi \equiv 0 \end{array} \right\}$$

$$= -\cancel{c_2} n\pi \cosh(n\pi) + \cancel{c_2} \sinh(n\pi) \Leftrightarrow$$

$$\Leftrightarrow 2n\pi \cosh(n\pi) = ((n\pi)^2 + 1)\sinh(n\pi) \Leftrightarrow$$

$$\Leftrightarrow \tanh(n\pi) = \frac{2n\pi}{1 + (n\pi)^2}$$

(check a plot of $\tanh(x) - \frac{2x}{1+x^2}$ in Wolfram) (google)

However, there is no integer $n \geq 1$ such that the above equation has a solution. That is, it is impossible to solve ② for $\mu_n = (n\pi)^2$, $n \geq 1$, unless $F(x) = 0$. That is, $\Phi \equiv 0$ is the only solution, so $\lambda = 0$ is not an eigenvalue.

$\boxed{P2}$ $$\left.\begin{array}{l} u_{tt} = \Delta u - \alpha u_t, \quad 0 < x < \pi, \; 0 < y < \pi, \\ u_x(0,y,t) = u_x(\pi,y,t) = 0 \\ u(x,0,t) = u(x,\pi,t) = 0 \end{array}\right\}$$

Part a)

$$E'(t) = \int_\Omega \left(2 u_t u_{tt} + 2 \nabla u \cdot \nabla u_t\right) dV = \left\{ \begin{array}{c}\text{integration by}\\ \text{parts}\end{array} \right\}$$

$$= 2\int_\Omega u_t u_{tt}\, dV - 2\int_\Omega \Delta u\, u_t\, dV + 2\int_{\partial\Omega} u_t \vec{n}\cdot\nabla u\, dS =$$

$$= 2\int_\Omega u_t(u_{tt} - \Delta u)\, dV + 2\int_0^\pi u_t(\cancel{x},0,t)(-u_y(x,0,t))\,dx +$$

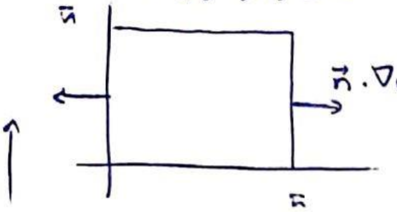
$$+ 2\int_0^\pi u_t(\pi,y,t)\, u_x(\pi,\cancel{y},t))\,dy + 2\int_0^\pi u_t(\cancel{x},\pi,t)\, u_y(x,\pi,t)\,dx +$$

$$+ 2\int_0^\pi u_t(0,y,t)(-u_x(0\cancel{,}y,t))\,dy =$$

$$= 2\int_\Omega u_t \overbrace{(u_{tt} - \Delta u)}^{\text{use the PDE}}\, dV = 2\int_\Omega u_t(-\alpha u_t)\, dV = -2\alpha\int_\Omega (u_t)^2 dV.$$

Notice that we have used that

$$u(x,\pi,t)=0 \Rightarrow \frac{\partial}{\partial t}\big(\overbrace{u(x,\pi,t)}^{=0\ \boxed{\text{for all } t}}\big) = u_t(x,\pi,t) = 0 \,//.$$



$$\vec{n}\cdot\nabla u(\pi,y,t) = u_x(\pi,y,t)$$

$u(x,0,t)=0 \Rightarrow u_t(x,0,t)=0$ because $u(x,0,t)=0$ for all $\underline{\underline{t}}$.

$\vec{n}\cdot\nabla u(0,\pi,t) = -u_x(0,y,t)$ (i.e., its constant in time).

In summary,

$$E'(t) = -2\alpha\int_\Omega |u_t|^2\,dV < 0 \Rightarrow \text{strictly decreasing.}$$

(unless $u_t \equiv 0$, that is, if the membrane is not moving).

Part b.

$$u(x,y,t) = \phi(x,y)h(t) \rightarrow \frac{h''(t)}{h(t)} + \alpha\frac{h'(t)}{h(t)} = \frac{\Delta\phi(x,y)}{\phi(x,y)} = -\lambda \Rightarrow$$

$$\Rightarrow \text{①}\ h''(t) + \alpha h'(t) + \lambda h(t) = 0$$

② $\Delta\phi + \lambda\phi = 0$ in $0<x<\pi,\ 0<y<\pi$,

$\phi(x,0) = \phi(x,\pi) = 0$

$\phi_x(0,y) = \phi_x(\pi,y) = 0$

We need to solve ② first. We separate x and y:

$$\Phi(x,y) = F(x)G(y) \rightarrow \frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)} - \lambda = -\mu \rightarrow$$

$$\rightarrow \text{(2.1)} \left. \begin{array}{l} F''(x) = -\mu F(x) \\ F'(0) = F'(\pi) = 0 \end{array} \right\} \qquad \text{(2.2)} \left. \begin{array}{l} G''(y) = -(\lambda-\mu)G(y) \\ G(0) = G(\pi) = 0 \end{array} \right\}$$

$$\text{(2.1)} \Rightarrow F_n(x) = A_n \cos(nx),\ \mu_n = n^2,\ n \geq 0.$$

Then,

$$\text{(2.2)} \left. \begin{array}{l} G_n''(y) = -(\lambda_n - n^2)G_n(y) \\ G_n(0) = G_n(\pi) = 0 \end{array} \right\} \Rightarrow$$

$$\Rightarrow \lambda_{n,m} - n^2 = m^2 \text{ with } m \geq 1 \Rightarrow \lambda_{n,m} = n^2 + m^2.$$

$$G_{n,m}(y) = B_{n,m} \sin(my).$$

- So $\Phi_{n,m}(x,y) = A_{n,m} \cos(nx)\sin(my)$, $n \geq 0$, $m \geq 1$.

With these values of $\lambda_{n,m} = n^2 + m^2$, $n \geq 0$, $m \geq 1$, we solve ①:

$$r = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\lambda_{n,m}}}{2} \quad \text{with } \alpha = \sqrt{5}.$$

Notice that for $\left.\begin{matrix} n=0 \\ m=1 \end{matrix}\right\} \to \alpha^2 - 4(n^2+m^2) = 1 > 0$.

For all other cases, $\alpha^2 - 4(n^2+m^2) < 0$.

So,

$$h_{0,1}(t) = A_{0,1} e^{\frac{-\sqrt{5}+1}{2}t} + B_{0,1} e^{\frac{-\sqrt{5}-1}{2}t},$$

$$h_{n,m}(t) = e^{-\frac{\sqrt{5}}{2}t}\left(A_{n,m}\cos(w_n t) + B_{n,m}\sin(w_n t)\right),$$

with $w_n = \frac{\sqrt{\alpha^2 - 4\lambda_{n,m}}}{2}$.

In summary, superposition principle gives

$$u(x,y,t) = \left(A_{0,1} e^{\frac{-\sqrt{5}+1}{2}t} + B_{0,1} e^{\frac{-\sqrt{5}-1}{2}t}\right)\sin(y) +$$

$$+ \sum_{n=0}^{\infty} \sum_{\substack{m=1 \\ n+m\geq 2}}^{\infty} e^{-\frac{\sqrt{5}}{2}t}\left(A_{n,m}\cos(w_n t) + B_{n,m}\sin(w_n t)\right)\cos(nx)\sin(my).$$

- We can check that $\lim_{t\to\infty} u(x,y,t) = 0$ (all the exponentials have negative exponent).

This coincides with Part a.): the energy decreases, the membrane evolves to its equilibrium $u \equiv 0$.

Part c)

$$u(x,y,0) = 0 = (A_{0,1} + B_{0,1})\sin(y) +$$

$$+ \sum_{n=0}^{\infty} \sum_{\substack{m=1 \\ n+m\geq 2}} A_{n,m}\cos(nx)\sin(my).$$

By orthogonality, $A_{0,1} + B_{0,1} = 0$, $A_{n,m} = 0$ for $n+m \geq 2$ $(n \geq 0, m \geq 1)$.

$$u_t(x,y,0) = f(x,y) =$$

$$= \left(A_{0,1}\frac{1-\sqrt{5}}{2} - \frac{1+\sqrt{5}}{2}B_{0,1} \right)\sin(y) +$$

$$+ \sum_{n=0}^{\infty} \sum_{\substack{m=1 \\ n+m\geq 2}}^{\infty} \frac{-\sqrt{5}}{2} B_{n,m} \sin(\cancel{\omega_n}\cdot 0)\ldots +$$

$$+ \sum_{n=0}^{\infty} \sum_{\substack{m=1 \\ n+m\geq 2}}^{\infty} B_{n,m}\,\omega_n \cos(nx)\sin(my) \Rightarrow$$

$$\Rightarrow \begin{cases} A_{0,1}\dfrac{1-\sqrt{5}}{2} - \dfrac{1+\sqrt{5}}{2}B_{0,1} = \dfrac{1}{\pi}\dfrac{2}{\pi}\displaystyle\int_0^{\pi}\int_0^{\pi} f(x,y)\,dx\,dy, \\[2ex] B_{n,m}\,\omega_n = \dfrac{4}{\pi^2}\displaystyle\int_0^{\pi}\int_0^{\pi} f(x,y)\cos(nx)\sin(my)\,dx\,dy \quad \begin{pmatrix} n\geq 0, m\geq 1 \\ n+m \geq 2 \end{pmatrix} \\[2ex] + \\ A_{0,1} + B_{0,1} = 0 \end{cases}$$

P3

Part a)

Let ϕ_1, ϕ_2 be eigenfunctions with $\lambda_1 \neq \lambda_2$. Then,

$$\left.\begin{aligned}\phi_1^{(IV)}(x) + \lambda_1\phi_1 e^x = 0\\ \phi_2^{(IV)}(x) + \lambda_2\phi_2 e^x = 0\end{aligned}\right| \begin{aligned}&\to \int_0^1 \phi_2\phi_1^{(IV)}dx = -\int_0^1 \lambda_1\phi_1\phi_2 e^x dx,\\ &\to \int_0^1 \phi_2^{(IV)}\phi_1 dx = -\int_0^1 \lambda_2\phi_2\phi_1 e^x dx\end{aligned}\Bigg| \Rightarrow$$

$$\Rightarrow (\lambda_1-\lambda_2)\int_0^1 \phi_1\phi_2 e^x dx = \int_0^1 \phi_2^{(IV)}\phi_1 dx - \int_0^1 \phi_2\phi_1^{(IV)} dx =$$

$$= -\int_0^1 \phi_2'''\phi_1' dx + \cancel{\phi_2'''\phi_1}\Big]_0^1 + \int_0^1 \phi_2'\phi_1''' dx - \cancel{\phi_2\phi_1'''}\Big]_0^1 =$$

$$= \int_0^1 \phi_2''\phi_1'' - \cancel{\phi_2''\phi_1'}\Big]_0^1 - \int_0^1 \phi_2''\phi_1'' + \cancel{\phi_2'\phi_1''}\Big]_0^1 = 0 \Rightarrow$$

$$\underset{\lambda_1\neq\lambda_2}{\Rightarrow} \int_0^1 \phi_1\phi_2 e^x dx = 0 \Rightarrow \phi_1, \phi_2 \text{ orthogonal with weight } e^x.$$

Part b.)

$$\phi^{(IV)} + \lambda e^x \phi = 0 \Rightarrow \phi\,\phi^{(IV)} + \lambda e^x \phi^2 = 0 \Rightarrow$$

$$\Rightarrow \int_0^1 \phi\,\phi^{(IV)}\,dx + \lambda \int_0^1 e^x \phi^2\,dx = 0 \Rightarrow$$

$$\Rightarrow \lambda = \frac{-\int_0^1 \phi\,\phi^{(IV)}\,dx}{\int_0^1 e^x \phi^2\,dx} = \frac{\int_0^1 \phi'\phi'''\,dx - \cancel{\phi\phi'''\Big]_0^1}}{\int_0^1 e^x \phi^2\,dx} =$$

$$= \frac{-\int_0^1 \phi''\phi''\,dx + \cancel{\phi'\phi''\Big]_0^1}}{\int_0^1 e^x \phi^2\,dx} = -\frac{\int_0^1 (\phi'')^2\,dx}{\int_0^1 e^x \phi^2\,dx} \le 0 \,/\!/.$$

Part c.)

If $\lambda = 0$ is eig., then $\phi''(x) = 0 \Rightarrow \phi(x) = c_1 x + c_2$.

BCs : $\left.\begin{array}{l}\phi(0) = c_2 = 0 \\ \phi(1) = c_1 = 0\end{array}\right\} \Rightarrow \phi \equiv 0 \;\; \times$ contradiction.

Thus, $\lambda = 0$ is not an eigenvalue.

Part d.)

Let's order the eigenvalues $\dots < \lambda_n < \dots < \lambda_2 < \lambda_1$.

Then,

$$\sin x \,\phi_m(x)\, e^x = \sum_{n\geq 1} A_n \phi_n(x)\phi_m(x) e^x \Rightarrow$$

$$\Rightarrow \int_0^1 \sin x \,\phi_m(x) e^x \, dx = \sum_{n\geq 1} A_n \int_0^1 \phi_n(x)\phi_m(x) e^x \, dx =$$

$$= A_m \int_0^1 \phi_m^2 e^x \, dx \Rightarrow$$

$$\Rightarrow A_m = \frac{\int_0^1 \sin x \,\phi_m(x) e^x \, dx}{\int_0^1 (\phi_m(x))^2 e^x \, dx}.$$

$\boxed{P4}$

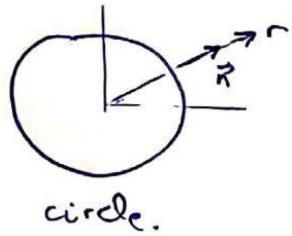
Part a)

$u(R,\theta,t) + u_r(R,\theta,t) = 0 \Rightarrow \phi(R,\theta)h(t) + \phi_r(R,\theta)h(t) = 0$

for all $t \geq 0 \Rightarrow$

$\Rightarrow \phi(R,\theta) + \phi_r(R,\theta) = 0 \Rightarrow \phi + \vec{n}\cdot\nabla\phi = 0$, since

$\vec{n}\cdot\nabla\phi = \phi_r$ on $\partial\Omega$.



outward normal direction $\equiv$ radial direction

Part b.)

$$\left.\begin{aligned}\Delta\phi + \lambda\phi &= 0\\ \phi &= -\vec{n}\cdot\nabla\phi\end{aligned}\right\}$$

Rayleigh quotient:

$$\lambda = \frac{-\int_{\partial\Omega}\phi\,\vec{n}\cdot\nabla\phi + \int_{\Omega}|\nabla\phi|^2}{\int_{\Omega}\phi^2} =$$

$$= \frac{\int_{\partial\Omega}\phi^2 + \int_{\Omega}|\nabla\phi|^2}{\int_{\Omega}\phi^2} \geq 0.$$

- If $\lambda = 0$, then $\int_{\partial\Omega}\phi^2 + \int_{\Omega}|\nabla\phi|^2 = 0 \Rightarrow \left\{\begin{aligned}&\phi \equiv 0 \text{ on } \partial\Omega\\ &\nabla\phi \equiv 0 \text{ in } \Omega\end{aligned}\right\} \Rightarrow$

$\Rightarrow \phi \equiv 0$ ↯ Thus, $\lambda > 0$.

Part c)

$$\left.\begin{aligned} & r^2 F''(r) + r F'(r) + (\lambda r^2 - n^2) F(r) = 0 \\ & \underline{BC}: \quad F(R) = -F'(R) \\ & \qquad\quad |F(0)| < \infty \end{aligned}\right\}$$

General solution to the ODE:

$$F(r) = c_1 J_n(\sqrt{\lambda} r) + c_2 Y_n(\sqrt{\lambda} r) \quad , \; n \geq 0 .$$

$$|F(0)| < \infty \Rightarrow c_2 = 0 \quad \text{since} \lim_{x \to 0} Y_n(x) = -\infty .$$

So, $F(r) = c_1 J_n(\sqrt{\lambda} r)$, with λ determine by

$$\| J_n(\sqrt{\lambda} R) = -\sqrt{\lambda} J_n'(\sqrt{\lambda} R) \| \Rightarrow \lambda_{n,m} /\!/ .$$ (order starting with $m=1$ for the first root).

Part d)

$$u(r, \theta, t) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} h_{n,m}(t) F_{n,m}(r) G_n(\theta) =$$

$$= \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} J_n(\sqrt{\lambda_{n,m}} r) \left(A_{n,m} \cos(n\theta) + B_{n,m} \sin(n\theta) \right) \cdot$$

$$\cdot \left(C_{n,m} \cos(\sqrt{\lambda_{n,m}} t) + D_{n,m} \sin(\sqrt{\lambda_{n,m}} t) \right) /\!/ .$$

Part e)

We assume that the solutions to the 2-d eigenvalue problem

$$\left.\begin{array}{ll}\Delta\phi + \lambda\phi & \text{in } \Omega \\ \phi = -\vec{n}\cdot\nabla\phi & \text{on } \partial\Omega\end{array}\right\} \text{ are orthogonal.}$$

Then,

$$u(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} h_{n,m}(t)\,\phi_{n,m}(r,\theta) =$$

$$= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} h_{n,m}(t)\left(A_{n,m}\phi^{c}_{n,m}(r,\theta) + B_{n,m}\phi^{s}_{n,m}(r,\theta)\right);$$

where we are denoting $\phi^{c}_{n,m} \equiv J_n(\sqrt{\lambda_{n,m}}\,r)\cos(n\theta)$,

$\phi^{s}_{n,m} \equiv J_n(\sqrt{\lambda_{n,m}}\,r)\sin(n\theta)$.

Then, since

$$\left.\begin{array}{l} u(r,\theta,0) = f(r,\theta) = \displaystyle\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} C_{n,m}\left(A_{n,m}\phi^{c}_{n,m} + B_{n,m}\phi^{s}_{n,m}\right), \\ u_t(r,\theta,0) = 0 = \displaystyle\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} D_{n,m}\sqrt{\lambda_{n,m}}\left(A_{n,m}\phi^{c}_{n,m} + B_{n,m}\phi^{s}_{n,m}\right)\end{array}\right| \Rightarrow$$

$$\Rightarrow \begin{array}{l} D_{n,m}A_{n,m} = 0 \\ D_{n,m}B_{n,m} = 0\end{array}, \quad A_{n,m}C_{n,m} = \frac{\int_\Omega f\,\phi^{c}_{n,m}\,dA}{\int_\Omega (\phi^{c}_{n,m})^2\,dA}$$

$$B_{n,m}C_{n,m} = \frac{\int_\Omega f\,\phi^{s}_{n,m}\,dA}{\int_\Omega (\phi^{s}_{n,m})^2\,dA}$$