$\boxed{\text{Midterm 2}}$: Solutions

$\boxed{\text{P1}}$

$$\left.\begin{array}{ll} \Delta\Phi + \lambda\Phi = 0 & \text{in } \Omega \\ \vec{n}\cdot\nabla\Phi = \Phi & \text{on } \partial\Omega \end{array}\right\} \quad \Omega \text{ disk}, R.$$

a)

$$\lambda = \frac{-\int_{\partial\Omega} \Phi \overbrace{\vec{n}\cdot\nabla\Phi}^{=\Phi \text{ on } \partial\Omega} d\ell + \int_{\Omega} |\nabla\Phi|^2 dA}{\int_{\Omega} \Phi^2 dA} = \frac{-\int_{\partial\Omega} \Phi^2 d\ell + \int_{\Omega} |\nabla\Phi|^2 dA}{\int_{\Omega} \Phi^2 dA}.$$

+1 +2 5/5

We cannot conclude anything about λ from this:

$-\int_{\partial\Omega} \Phi^2 d\ell$ is negative, $\int_{\Omega} |\nabla\Phi|^2 dA$ is positive, we don't know which one is bigger.

+2

b) This is an Euler-Cauchy ODE:

$\underline{n>0}$: $f(x) = c_1 x^n + c_2 x^{-n}$ $\left(x^p \to x^2 p(p-1)x^{p-2} + xpx^{p-1} - n^2x^p = 0 \Rightarrow p^2 - p + p - n^2 = 0 \Rightarrow p = \pm n \right).$

+3

$\underline{n=0}$: $x^2 f''(x) + x f'(x) = 0$ 5/5

$\hookrightarrow x \frac{d}{dx}\left(x f'(x)\right) = 0 \Rightarrow x f'(x) = c_1 \Rightarrow f(x) = c_1 \log x + c_2.$ //.

+2

c) $\phi(r,\theta) = F(r)\,G(\theta)$, $\lambda = 0$.

$$\left.\begin{aligned} \Delta\phi &= 0 \text{ in } \Omega \\ \phi &= \vec{n}\cdot\nabla\phi \text{ on } \partial\Omega \end{aligned}\right\} \quad \Delta\phi(r,\theta) = \frac{1}{r}\frac{d}{dr}\left(r\frac{\partial\phi}{\partial r}(r,\theta)\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial \theta^2}(r,\theta)$$

$$\frac{G(\theta)}{r}\frac{d}{dr}\left(rF'(r)\right) + \frac{F(r)}{r^2}\,G''(\theta) = 0 \Rightarrow$$

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$$\Rightarrow \frac{r}{F(r)}\frac{d}{dr}\left(rF'(r)\right) = \frac{-G''(\theta)}{G(\theta)} = \mu \Rightarrow$$

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$$\Rightarrow \text{①}\; G''(\theta) = -\mu\, G(\theta) \;, \quad \text{②}\; r\frac{d}{dr}\left(rF'(r)\right) = \mu F(r)$$

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BC:

$$\vec{n}\cdot\nabla\phi(R,\theta) = \frac{\partial}{\partial r}\left(\phi(R,\theta)\right) = F'(R)\,G(\theta) \overset{\text{for } \theta\in[-\pi,\pi]}{=} \phi(R,\theta) = F(R)\,G(\theta) \Leftrightarrow$$

$$\Leftrightarrow F(R) = F'(R)$$

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Also, $|F(0)| < \infty$, $G(-\pi) = G(\pi)$, $G'(-\pi) = G'(\pi)$.

+1 +1

d) $\lambda = 0$ is an eigenvalue if there exists ϕ (not equal to the zero function) solving

$$\left.\begin{aligned} \Delta\phi &= 0 \text{ in } \Omega \\ \phi &= \vec{n}\cdot\nabla\phi \text{ on } \partial\Omega \end{aligned}\right\}$$

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Let's find the solutions by separation of variables:

(part c)) $\Longrightarrow$ ① $G(\theta) = c_1 \cos(n\theta) + c_2 \sin(n\theta)$,

$$\mu_n = n^2, \quad n \geq 0$$

② $r^2 F''(r) + r F'(r) - n^2 F(r) = 0 \quad , \quad 0 < r < R$

$F'(R) = F(R) \,, |F(0)| < +\infty$

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(Using part b))

$\hookrightarrow$ $\underline{n=0}$: $F(r) = c_1 \log(r) + c_2 \rightarrow$ $|F(0)| < \infty \rightarrow c_1 = 0$

$\hookrightarrow F(r) = c_2$

$\hookrightarrow F'(R) = 0 \neq F(R) = c_2$!

$\hookrightarrow$ $\underline{n>0}$: $F(r) = c_1 r^n + c_2 \frac{1}{r^n}$

$|F(0)| < \infty \Rightarrow c_2 = 0 \Rightarrow F(r) = c_1 r^n \rightsquigarrow F'(r) = c_1 n r^{n-1}$

$F'(R) = c_1 n R^{n-1}$

$F(R) = c_1 R^n$

$F'(R) = F(R) \Longleftrightarrow n c_1 R^{n-1} = c_1 R^n \Longleftrightarrow$

+1

$\Longleftrightarrow$

- $c_1 = 0 \rightarrow F \equiv 0 \rightarrow \Phi \equiv 0$
- $n R^{n-1} = R^n \Longleftrightarrow \frac{n}{R} = 1 \Longleftrightarrow n = R$. +1
- $R = 0$ (the Ω is not a disk)

So what's the conclusion ?

Notice that n is a positive integer: $n = 1, 2, \ldots$

We have shown that

$F(r) = c_1 r^n$ solves the radial problem if $n = R$.

This is only possible if $R = 1, 2, \ldots$.

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In summary:

- $R \neq 1, 2, \ldots \Rightarrow$ No solution $\phi \neq 0 \Rightarrow \lambda = 0$ is not an eigenvalue.
- $R = 1, 2, \ldots \Rightarrow \phi(r, \theta) = (c_1 \cos(R\theta) + c_2 \sin(R\theta))\, r^R$ are eigenfunctions corresponding to $\lambda = 0$.

$\boxed{P2}$

a)

$$\left.\begin{array}{l} 4(\phi_{n,m})_{xx} + (\phi_{n,m})_{yy} + \lambda\phi_{n,m} = 0 \quad \text{in } \Omega \\ \phi_{n,m} = 0 \quad \text{on } \partial\Omega \end{array}\right\}$$

$\tilde{\Delta}\phi_{n,m} = 4(\phi_{n,m})_{xx} + (\phi_{n,m})_{yy}$ in ... $\phi_{n,m}(x,y) = \sin\left(\frac{n\pi x}{2}\right)\sin(m\pi y) - \sin\left(\frac{m\pi x}{2}\right)\sin(n\pi y)$,

- $\partial_x \phi_{n,m}(x,y) = \frac{1}{2}n\pi\cos\left(\frac{n\pi x}{2}\right)\sin(m\pi y) - \frac{m\pi}{2}\cos\left(\frac{m\pi x}{2}\right)\sin(n\pi y) \rightarrow$

$\partial_{xx}\phi_{n,m}(x,y) = \frac{-1}{4}n^2\pi^2\sin\left(\frac{n\pi x}{2}\right)\sin(m\pi y) + \frac{m^2\pi^2}{4}\sin\left(\frac{m\pi x}{2}\right)\sin(n\pi y)$

Analogously,

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$\partial_{yy}\phi_{n,m}(x,y) = -m^2\pi^2\sin\left(\frac{n\pi x}{2}\right)\sin(m\pi y) + n^2\pi^2\sin\left(\frac{m\pi x}{2}\right)\sin(n\pi y)$

Thus,

$$\tilde{\Delta}\phi_{n,m}(x,y) = -n^2\pi^2\overbrace{\left(\sin\left(\frac{n\pi x}{2}\right)\sin(m\pi y) - \sin\left(\frac{m\pi x}{2}\right)\sin(n\pi y)\right)}^{=\phi_{n,m}(x,y)} +$$

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$$-m^2\pi^2\left(\sin\left(\frac{n\pi x}{2}\right)\sin(m\pi y) - \sin\left(\frac{m\pi x}{2}\right)\sin(n\pi y)\right) =$$

$$= -(n^2+m^2)\pi^2\,\phi_{n,m}(x,y) \Rightarrow$$

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+4

$$\Rightarrow \lambda_{n,m} = -\pi^2(n^2+m^2)$$

b)

$u(x,y,t) = \phi(x,y)\, h(t)$ +2

$$\frac{h''(t)}{h(t)} = \frac{\Delta\phi(x,y)}{\phi(x,y)} \overset{+2}{=} \lambda \Rightarrow \begin{cases} h'' = -\lambda h & +2 \quad (\lambda < 0) \\ \left. \begin{array}{l} \tilde{\Delta}\phi = \lambda\phi \text{ in } \Omega \\ \phi = 0 \text{ on } \partial\Omega \end{array} \right\} \end{cases}$$

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Thus,

$$u(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} h_{n,m}(t)\, \Phi_{n,m}(x,y), \quad \text{with}$$

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$$h_{n,m}(t) = A_{n,m} \cos(\sqrt{-\lambda_{n,m}}\, t) + B_{n,m} \sin(\sqrt{-\lambda_{n,m}}\, t) \quad +2$$

c)

+1

$$u(x,y,0) = f(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \overbrace{h_{n,m}(0)}^{A_{n,m}} \phi_{n,m}(x,y) \Rightarrow$$

+ 3.5

$$\Rightarrow \int_\Omega f \phi_{k,e}\, dA = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{n,m} \int_\Omega \phi_{n,m} \phi_{k,e}\, dA = A_{k,e} \overbrace{\int_\Omega (\phi_{k,e})^2}^{=c}$$

+1

$$u_t(x,y,0) = g(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sqrt{-\lambda_{n,m}}\, B_{n,m}\, \phi_{n,m}(x,y) \Rightarrow$$

+3.5

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$$\Rightarrow B_{n,m} \sqrt{-\lambda_{n,m}}\, c = \int_\Omega g\, \phi_{n,m}\, dA.$$

$\boxed{\text{P3}}$

a) $\left.\begin{array}{l} \phi'' + (2-4x)\phi' = -\lambda\phi \\ \phi(0) = \phi(1) = 0 \end{array}\right\}$

$H(x)\phi''(x) + H(x)(2-4x)\phi'(x) + \lambda H(x)\phi(x) = 0$

$\hookrightarrow \frac{d}{dx}\left(H(x)\phi'(x)\right) + \lambda H(x)\phi(x) = 0$ if we define $H(x)$ by

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$\| H'(x) = H(x)(2-4x)$.

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That is, $\int \frac{dH}{H} = \int (2-4x)\,dx \rightarrow \log H = 2x - 2x^2 + C \rightarrow$

+ 4

$\rightarrow H(x) = c\, e^{2x-2x^2} \rightsquigarrow$ Take $H(x) = e^{2x-2x^2}$.

Then, we have an S-L ODE,

+1

$\frac{d}{dx}\left(H(x)\phi'(x)\right) + \lambda H(x)\phi(x) = 0$ with $\left\{\begin{array}{l} p(x) = H(x) = \sigma(x) \\ q(x) = 0 \end{array}\right.$

b) $\left.\begin{aligned}\phi'(x) &= 1-2x \\ \phi''(x) &= -2\end{aligned}\right\}$

$\phi'' + (2-4x)\phi' = -2 + (2-4x)(1-2x) = -\lambda(x-x^2) \Leftrightarrow$

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$\Leftrightarrow \left.\begin{aligned} 8x^2 &= \lambda x^2 \\ -8x &= -\lambda x \\ -2+2 &= 0 \end{aligned}\right\} \Leftrightarrow \boxed{\lambda = 8}$ ~ with eigenfunction $\phi(x) = x(x-1)$.

In part a) we found this is a regular S-L problem:

→ Dirichlet BC ✓,

→ $p(x) = \sigma(x) = e^{2x-2x^2} > 0$,

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→ p, q, σ continuous everywhere.

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Thus, $\phi_n(x)$ has exactly $n-1$ roots in $(a,b) = (0,1)$.

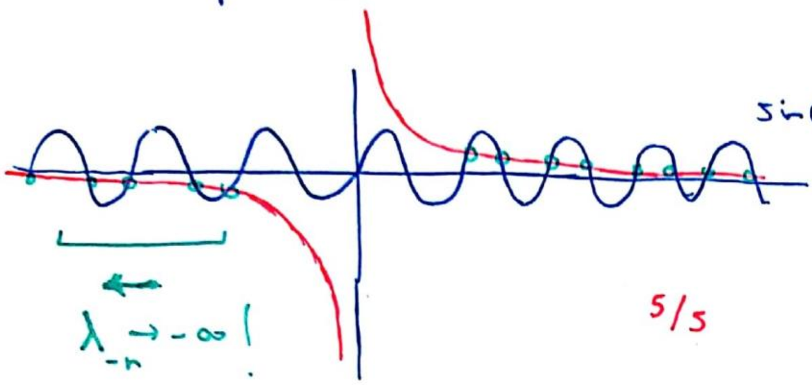
We see that $\phi(x) = x(x-1)$ has zero roots on $(0,1)$,

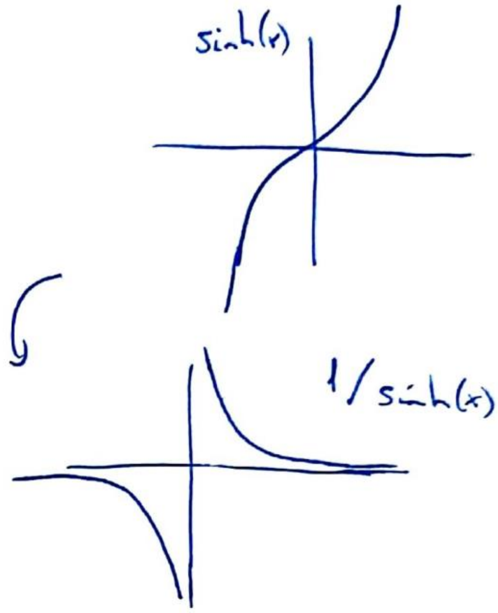
so $n = 1$.

c)

$$\sinh\lambda \sin\lambda = 1 \iff \sin\lambda = \frac{1}{\sinh\lambda}.$$

Graphically, we can see that





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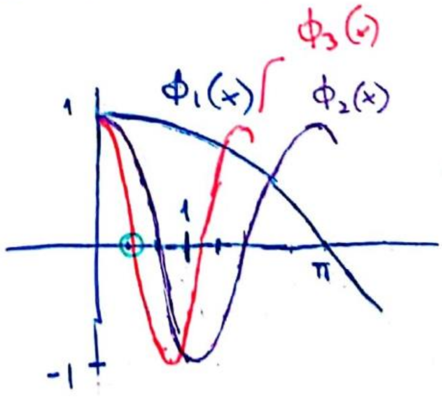
There infinitely many roots to the left and right. In particular, there is not a smallest eigenvalue.

So this cannot be a regular S-L problem.

d) $\lambda_n = n^2, n \geq 1, \ \phi_n(x) = \cos\left(\lambda_n \frac{x}{2}\right), \ 0 < x < 1.$

↳ $\phi_n(x) = \cos\left(n^2 \frac{x}{2}\right).$ 5/5



Let $n = 1: \ \phi_1(x) = \cos(x/2) \longrightarrow$

$n = 2: \dots$

$n = 3: \ \phi_3(x) = \cos\left(\frac{9}{2}x\right)$

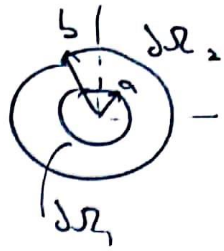
↳ has one root in $(0,1)$!

(instead of $n-1=2$) ↳ So not a regular S-L problem.

$\boxed{P4}$

$$\left.\begin{aligned} &u_{tt} = \Delta u \quad \text{in } \Omega \\ &u = 0 \quad \text{on } \partial\Omega_1 \\ &\vec{n}\cdot\nabla u = 0 \quad \text{on } \partial\Omega_2 \\ &u|_{t=0} = f, \quad u_t|_{t=0} = 0 \end{aligned}\right\}$$

Ω annulus



a) $u(a,\theta,t) = 0 \to \phi(a,\theta) = 0 \overset{+1}{\Rightarrow} F(a) = 0$

$\vec{n}\cdot\nabla u(b,\theta,t) = 0 \to \dfrac{\partial\phi}{\partial r}(b,\theta) = 0 \overset{+2}{\Rightarrow} F'(b) = 0$. 3/3

b) $$\lambda = \frac{-\int_{\partial\Omega}\cancel{\phi\,\vec{n}\cdot\nabla\phi\, dl} + \int_\Omega |\nabla\phi|^2 dA}{\int_\Omega \phi^2\, dA} = \frac{\int_\Omega |\nabla\phi|^2 dA}{\int_\Omega \phi^2\, dA} \geq 0.$$ +2

$$\int_{\partial\Omega} \phi\,\vec{n}\cdot\nabla\phi\, dl = \int_{\partial\Omega_1} \cancel{\phi}\,\vec{n}\cdot\nabla\phi\, dl + \int_{\partial\Omega_2} \phi\,\cancel{\vec{n}\cdot\nabla\phi}\, dl = 0.$$ 4/4

- Now, if $\lambda = 0$, then $\int_\Omega |\nabla\phi|^2 dA = 0 \to \nabla\phi = 0$ in $\Omega \to$

+2

$\Rightarrow \phi \equiv$ constant.

But since $\phi \equiv 0$ on $\partial\Omega_1$ $\Big| \to \phi \equiv 0$ in $\Omega \to$ not eigenfunction.

Thus $\lambda = 0$ is not an eigenvalue.

c)

$$\begin{cases} r^2 F''(r) + rF'(r) + (\lambda r^2 - n^2)F(r) = 0 \quad , \; n \geq 0 \text{ fixed.} \\ F(a) = 0, \; F'(b) = 0 \end{cases}$$

Bessel ODE order n:

$$F(r) = c_1 J_n(\sqrt{\lambda} r) + c_2 Y_n(\sqrt{\lambda} r)$$

$$F(a) = c_1 J_n(\sqrt{\lambda} a) + c_2 Y_n(\sqrt{\lambda} a) = 0$$ + 2

$$F'(b) = c_1 \sqrt{\lambda} J_n'(\sqrt{\lambda} b) + c_2 Y_n'(\sqrt{\lambda} b)\sqrt{\lambda} = 0$$ $\Leftrightarrow$ $(\lambda > 0)$

$$\Leftrightarrow \underbrace{\begin{bmatrix} J_n(\sqrt{\lambda} a) & Y_n(\sqrt{\lambda} a) \\ J_n'(\sqrt{\lambda} b) & Y_n'(\sqrt{\lambda} b) \end{bmatrix}}_{A} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0$$

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If $\det(A) = 0$, the there are infinite solutions c_1, c_2 (otherwise $c_1 = c_2 = 0$). So the equation that defines the λ's is

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$$\boxed{J_n(\sqrt{\lambda} a) Y_n'(\sqrt{\lambda} b) = J_n'(\sqrt{\lambda} b) Y_n(\sqrt{\lambda} a)}$$

d)

$u(r,\theta,t) =$

$$= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \left(A_{n,m}\cos(\sqrt{\lambda_{n,m}}\,t) + B_{n,m}\sin(\sqrt{\lambda_{n,m}}\,t)\right)\left(C_{n,m}\cos(n\theta) + D_{n,m}\sin(n\theta)\right)\cdot$$

$$\left(Z_{n,m} J_n(\sqrt{\lambda_{n,m}}\,r) + W_{n,m} Y_n(\sqrt{\lambda_{n,m}}\,r)\right).$$

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$\rightarrow A_{n,m}, B_{n,m}, C_{n,m}, D_{n,m}, Z_{n,m}, W_{n,m}$ constants;

= (Not required)

IC: $u(r,\theta,0) = f(r,\theta) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{n,m}\left(C_{n,m}\cos(n\theta) + D_{n,m}\sin(n\theta)\right)\cdot$

$A_{n,m} C_{n,m}, D_{n,m}$ ... $\Leftarrow$ $\left(Z_{n,m} J_n Y_n(\sqrt{\lambda_{n,m}}\,r) + W_{n,m} Y_n(\sqrt{\lambda_{n,m}}\,r)\right).$

$Z_{n,m}, W_{n,m}$ //.

$$u_t(r,\theta,0) = 0 = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} B_{n,m}\sqrt{\lambda_{n,m}}\left(C_{n,m}\cos(n\theta) + D_{n,m}\sin(n\theta)\right)$$

$$\left(Z_{n,m} J_n(\sqrt{\lambda_{n,m}}\,r) + W_{n,m} Y_n(\sqrt{\lambda_{n,m}}\,r)\right)$$

$\Rightarrow B_{n,m} = 0.$