Midterm 2: Solutions.

P1

a) $\phi h'' = \phi'' h - 3\phi h' \Rightarrow \frac{h''}{h} = \frac{\phi''}{\phi} - 3\frac{h'}{h} \Rightarrow$

$$\Rightarrow \frac{h''}{h} + 3\frac{h'}{h} = \frac{\phi''}{\phi} = -\lambda \Rightarrow \begin{cases} \phi''(x) = -\lambda\phi(x), \\ h''(t) + 3h'(t) = -\lambda h(t). \end{cases}$$

4/4

+2

BC: $u(0,t) = \phi(0)h(t) = 0$ for $t \geq 0 \Rightarrow \phi(0) = 0$.

$u_x(\pi,t) = \phi'(\pi)h(t) = 0$ for $t \geq 0 \Rightarrow \phi'(\pi) = 0$.

+2

4/4 b) $\phi'' = -\lambda\phi \xrightarrow{(*)} \int_0^\pi \phi''(x)\phi(x)\,dx = -\lambda \|\phi\|_{L^2}^2 \Rightarrow$

$\phi(0) = \phi'(\pi) = 0$

$$\Rightarrow \lambda = +\frac{1}{\|\phi\|_{L^2}^2}\int_0^\pi (\phi'(x))^2 dx - \frac{1}{\|\phi\|_{L^2}^2}\Big[\cancel{\phi'(x)\phi(x)}\Big]_0^\pi = \frac{1}{\|\phi\|_{L^2}^2}\int_0^\pi (\phi'(x))^2 dx \geq 0.$$

+1

If $\lambda = 0$, then $\phi'(x) \equiv 0 \Rightarrow \phi(x) = c$.

+1

Since $\phi(0) = 0$, this implies $\phi \equiv 0$ $\not\Rightarrow$ (ϕ is eigenfunction).

Thus $\lambda \neq 0$. That is, $\underline{\lambda > 0}$.

[(*) $\lambda \in \mathbb{R}$ because the BC are symmetric]

Let $\phi_1, \phi_2 \sim \lambda_1 \neq \lambda_2$. Then,

$$\left.\begin{array}{l} \langle \phi_1'', \phi_2 \rangle = -\lambda_1 \langle \phi_1, \phi_2 \rangle \\ \langle \phi_2'', \phi_1 \rangle = -\lambda_2 \langle \phi_1, \phi_2 \rangle \end{array}\right\} \rightarrow$$

$$\rightarrow (\lambda_1 - \lambda_2) \langle \phi_1, \phi_2 \rangle = \langle \phi_2'', \phi_1 \rangle - \langle \phi_1'', \phi_2 \rangle =$$

$$= \cancel{-\langle \phi_2', \phi_1' \rangle} + \phi_2' \phi_1 \Big]_0^{\pi} + \cancel{\langle \phi_1', \phi_2' \rangle} - \phi_1' \phi_2 \Big]_0^{\pi} = 0 \rightarrow$$

$\lambda_1 \neq \lambda_2$.

$$\rightarrow \langle \phi_1, \phi_2 \rangle = 0 .$$ +2

c) We know from b.) that $\lambda > 0$, therefore,

4/4 $\phi(x) = c_1 \sin(\sqrt{\lambda}\, x) + c_2 \cos(\sqrt{\lambda}\, x)$. +1

$\phi(0) = c_2 = 0$

$\phi'(\pi) = c_1 \sqrt{\lambda} \cos(\sqrt{\lambda}\, \pi) = 0 \rightarrow \sqrt{\lambda}\, \pi = (2n-1)\frac{\pi}{2}$, $n \geq 1$.

Thus,
$$\left.\begin{array}{l} \lambda_n = \left(\frac{2n-1}{2}\right)^2 \\ \phi_n(x) = \sin\left(\frac{2n-1}{2} x\right) \end{array}\right\} n \geq 1.$$ +3

d) $h''(t) + 3h(t) + \left(\frac{2n-1}{2}\right)^2 h(t) = 0$

4/4

Characteristic polynomial: $r^2 + 3r + \left(\frac{2n-1}{2}\right)^2 = 0$,

$$r = \frac{-3 \pm \sqrt{9 - 4\left(\frac{2n-1}{2}\right)^2}}{2}$$

$\underline{n=1}$: $r = \frac{-3 \pm \sqrt{9-1}}{2} = \frac{-3 \pm \sqrt{8}}{2} \rightarrow$

$\rightarrow h_1(t) = a_1 e^{\frac{-3+\sqrt{8}}{2} t} + b_1 e^{\frac{-3-\sqrt{8}}{2} t}$, +1

$\underline{n=2}$: $r = \frac{-3 \pm \sqrt{9 - 4\frac{9}{4}}}{2} = \frac{-3}{2} \rightarrow$

+1

$\rightarrow h_2(t) = a_2 e^{-\frac{3}{2}t} + b_2 t e^{-\frac{3}{2}t}$,

+2

$\underline{n \geq 3}$: $r = \frac{-3 \pm \sqrt{9 - 4\left(\frac{2n-1}{2}\right)^2}}{2} = \frac{-3}{2} \pm i\omega_n$, $\omega_n = \sqrt{4\left(\frac{2n-1}{2}\right)^2 - 9} \in \mathbb{R}$.

$\rightarrow h_n(t) = e^{-\frac{3}{2}t}\left(a_n \cos(\omega_n t) + b_n \sin(\omega_n t)\right)$.

e) For example,

4/4

$u_1(x,t) = h_1(t)\phi_1(t) = \left(a_1 e^{\frac{-3+\sqrt{8}}{2}t} + b_1 e^{\frac{-3-\sqrt{8}}{2}t}\right)\sin\left(\frac{x}{2}\right)$,

for any $a_1, b_1 \in \mathbb{R}$.

5/5 f)

$$u(x,t) = \left(a_1 e^{\frac{-3+\sqrt{8}}{2}t} + b_1 e^{\frac{-3-\sqrt{8}}{2}t}\right)\sin(x/2) +$$

$$+ \left(a_2 e^{-3/2 t} + b_2 t e^{-3/2 t}\right)\sin(3x/2) +$$

+3 $$+ \sum_{n\geq 3}\left(a_n\cos(w_n t) + b_n\sin(w_n t)\right)e^{-\frac{3}{2}t}\sin\left(\frac{2n-1}{2}x\right).$$

I.C.:

$$u(x,0) = 2\pi x - x^2 = (a_1+b_1)\sin(x/2) + a_2\sin(3x/2) +$$

$$+ \sum_{n\geq 3} a_n \sin\left(\frac{2n-1}{2}x\right),$$

$$u_t(x,0) = 0 = \left(\frac{-3+\sqrt{8}}{2}a_1 - \frac{3+\sqrt{8}}{2}b_1\right)\sin(x/2) + \left(\frac{-3}{2}a_2 + b_2\right)\sin(3x/2) +$$

+2 $$+ \sum_{n\geq 3} b_n w_n \sin\left(\frac{2n-1}{2}x\right) + \sum_{n\geq 3}\frac{-3}{2}a_n\sin\left(\frac{2n-1}{2}x\right).$$

Using the orthogonality of $\{\sin(\frac{2n-1}{2}x)\}$ and that $\|\sin(\frac{2n-1}{2}x)\|_{L^2}^2 = \frac{\pi}{2}$,

we find that

$$a_1+b_1 = \frac{2}{\pi}\int_0^{\pi}(2\pi x - x^2)\sin(x/2)\,dx, \quad \frac{-3+\sqrt{8}}{2}a_1 - \frac{3+\sqrt{8}}{2}b_1 = 0,$$

$$a_2 = \frac{2}{\pi}\int_0^{\pi}(2\pi x - x^2)\sin(3x/2)\,dx, \quad \frac{-3}{2}a_2 + b_2 = 0,$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}(2\pi x - x^2)\sin\left(\frac{2n-1}{2}x\right)dx, \quad b_n w_n - \frac{3}{2}a_n = 0$$

$$\left.\begin{array}{l} a_1 + b_1 = \frac{4}{\pi}(8 - 4\pi + \pi^2) \\ (-3+\sqrt{8})a_1 - (3+\sqrt{8})b_1 = 0 \end{array}\right\} \to a_1, b_1 //.$$

(wolfram).

$$a_2 = -\frac{4}{27\pi}(8 + 12\pi + 9\pi^2), \quad b_2 = \frac{3}{2}a_2 //.$$

$$a_n = -\frac{4(4\pi(2n-1) + (\pi^2(1-2n)^2 + 8)(-1)^n}{\pi(2n-1)^3}, \quad b_n = \frac{3}{2w_n}a_n /$$

$\boxed{P2}$

5/5

a) $$C_{2n-1} = \frac{1}{\|\phi_{2n-1}\|_{L^2}} \int_0^2 f(x)\cos\left(\frac{(2n-1)\pi x}{2}\right) dx, \quad \|\phi_{2n-1}\|_{L^2}^2 = 1.$$

We see that f is "even" across $x=1$, Φ_n is "odd" across $x=1$. To see it better, we change variables:

$y = x - 1 \to$

$$C_{2n-1} = \int_{-1}^{1} f(1+y)\cos\left(\frac{(2n-1)\pi}{2}(1+y)\right) dy =$$

$$= \int_{-1}^{1} \frac{1}{|y|^{2/5}}\Big(\underbrace{\cos\left(\frac{(2n-1)}{2}\pi\right)}_{=0}\cos(y) - \sin\left(\frac{2n-1}{2}\pi\right)\sin(y)\Big) dy =$$

$$= (-1)^n \int_{-1}^{1} \underbrace{\frac{\sin(y)}{|y|^{2/5}}}_{\text{odd}} = 0 //.$$

5/5

b) $\int_0^2 (f(x))^2\,dx = \int_0^2 \frac{dx}{|1-x|^{4/5}} = \int_0^1 \frac{dx}{(1-x)^{4/5}} + \int_1^2 \frac{dx}{(x-1)^{4/5}} =$

$= \left.\frac{-(1-x)^{1/5}}{1/5}\right]_0^1 + \left.\frac{(x-1)^{1/5}}{1/5}\right]_1^2 = \frac{1}{1/5} + \frac{1}{1/5} = 10 \rightarrow$

$\rightarrow \|f\|_{L^2} = \sqrt{10}$.

5/5

c) The set $\{\cos(\frac{n\pi x}{2})\}_{n\geq 0}$ is complete, $f \in L^2$, thus Parseval equality holds: +2

$$c_0 = \frac{5}{3}$$

$$\|f\|_{L^2}^2 = 10 = \sum_{n=0}^{\infty} c_n^2 \underbrace{\|\phi_n\|_{L^2}^2}_{=\begin{cases}1, & n\geq 1\\ 2, & n=0\end{cases}} = 2c_0^2 + \sum_{n\geq 1} c_n^2 \Rightarrow \sum_{n\geq 0} c_n^2 = 10 - \frac{25}{9} = \frac{65}{9}.$$

+3

5/5

d) Assume that $\sum_{n=0} |c_n| < +\infty$. Then, the series

$\sum_{n=0}^{\infty} c_n \cos(\frac{n\pi x}{2})$ converges uniformly to $f(x)$ in $(0,2)$. (*)

But $\cos(\frac{n\pi x}{2})$ is continuous on $(0,2)$ for every $n \geq 0$,

so $f(x)$ should be continuous. However, it has a discontinuity at $x=1$.

Therefore, $\sum_{n=0} |c_n| = +\infty$.

(+2.5 if "intuitive proofs" "formal's proofs")

((*) since we already know that it converges to $f(x)$ in $L^2(0,2)$).

e) (Check pages -4;-5; lect. notes 13).

5/5

$$\|f - S_6(f)\|_{L^2}^2 = \|f\|_{L^2}^2 - \sum_{n=0}^{6} c_n^2 \|\phi_n\|_{L^2}^2 =$$

$$= 10 - 2\left(\frac{5}{3}\right)^2 - \sum_{n=1}^{6} c_n^2 \approx 10 - 6.9854 = 3.0146,$$

$$\rightarrow 100 \cdot \frac{\sqrt{3.0146}}{\sqrt{10}} = 54.91\ \%\ \text{error (in } L^2).$$

For $N = 8$: $\|f - S_8(f)\|_{L^2}^2 \approx 10 - 7.1283 = 2.8717$

$N = 10$: $\|f - S_{10}(f)\|_{L^2}^2 \approx 10 - 7.2432 = 2.7568$

$\%\,\text{error}_{N=8} \approx 53.59\ \%$, $\%\,\text{error}_{N=10} \approx 52.51\ \%$.

$54.86 - 53.57 = 1.29$

$53.57 - 52.44 = 1.13$

P3

5/5 a) $c_n = \frac{1}{\|\phi_n\|_{L^2}^2} \int_0^{\pi} f(x) \cos\left(\frac{2n-1}{2}x\right) dx = \frac{2}{\pi} \int_0^{\pi} f(x) \cos\left(\frac{2n-1}{2}x\right) dx.$

5/5 b) $S_N(f)(x) = \sum_{n=1}^{N} \left(\frac{2}{\pi} \int_0^{\pi} f(y) \cos\left(\frac{2n-1}{2}y\right) dy \right) \cos\left(\frac{2n-1}{2}x\right) =$

$$= \frac{2}{\pi} \int_0^{\pi} f(y) \left(\sum_{n=1}^{N} \cos\left(\frac{2n-1}{2}y\right) \cos\left(\frac{2n-1}{2}x\right) \right) dy =$$

$$= \frac{1}{\pi} \int_0^{\pi} f(y) \sum_{n=1}^{N} \left(\cos\left(\frac{2n-1}{2}(x-y)\right) + \cos\left(\frac{2n-1}{2}(x+y)\right) \right) dy =$$

$$= \frac{1}{\pi} \int_0^{\pi} \left(K_N(x-y) + K_N(x+y) \right) f(y) dy \quad \text{if} \quad K_N(x) = \sum_{n=1}^{N} \cos\left(\frac{2n-1}{2}x\right).$$

5/5 c)

$$K_N(x) = \sum_{n=1}^{N} \frac{e^{i\frac{2n-1}{2}x} - e^{-i\frac{2n-1}{2}x}}{2} = \frac{1}{2} \sum_{n=-N+1}^{N} e^{i\frac{2n-1}{2}x} =$$

$$= \frac{1}{2} \frac{e^{-i\frac{2N-1}{2}x} - e^{i\frac{2N+1}{2}x}}{1-e^{ix}} = \frac{1}{2} \frac{e^{-iNx} - e^{iNx}}{e^{-ix/2} - e^{ix/2}} = \frac{1}{2} \frac{\sin(Nx)}{\sin(x/2)}$$

5/5

d) $\frac{1}{\pi}\int_0^{\pi} K_N(y)\,dy = \frac{1}{2} + a_N \Rightarrow$

$$\Rightarrow \frac{1}{2} = \frac{1}{\pi}\int_0^{\pi} K_N(y)\,dy - a_N \Rightarrow \frac{1}{2}f(x) = \frac{1}{\pi}\int_0^{\pi} K_N(y) f(x)\,dy - a_N f(x).$$

(1) $\Rightarrow$

$$\rightarrow S_N(f)(x) - f(x) = \frac{1}{\pi}\int_0^{\pi} K_N(y) f(x-y)\,dy + \frac{1}{\pi}\int_0^{\pi} K_N(y) f(x+y)\,dy + - \frac{1}{2}f(x) - \frac{1}{2}f(x) =$$

$$= \frac{1}{\pi}\int_0^{\pi} K_N(y)\big(f(x-y) - f(x)\big)\,dy + a_N f(x) + $$

$$+ \frac{1}{\pi}\int_0^{\pi} K_N(y)\big(f(x+y) - f(x)\big)\,dy + a_N f(x) \,//.$$

5/5

e) $$S_N(f)(x) - f(x) = \frac{1}{2\pi}\int_0^{\pi} \sin(Ny)\,\frac{f(x-y)-f(x)}{\sin(y/2)}\,dy + $$

$$+ \frac{1}{2\pi}\int_0^{\pi} \sin(Ny)\,\frac{f(x+y)-f(x)}{\sin(y/2)}\,dy + 2a_N f(x).$$

- $\{\sin(Ny)\}_{N\geq 1}$ orthogonal.
- $f \in C^1[0,2] \Rightarrow g(y) = \dfrac{f(x-y)-f(x)}{\sin(y/2)} \in L^2$

$\Big\} \Rightarrow \langle \sin(Ny), g(y)\rangle \to 0$ as $N \to \infty$.

pages -8-,...,-10- lect notes 14.

The same idea shows that $h(y) = \frac{f(x+y)-f(x)}{\sin(y/2)} \in L^2$.

Finally, since f continuous on $[0,\pi]$, f is bounded,

so $\lim_{N\to\infty} |a_N f(x)| \leq \left(\lim_{N\to\infty} a_N\right) \max_{x\in[0,\pi]} f(x) = 0$.

In conclusion, $\lim_{N\to\infty} |S_N(f)(x) - f(x)| = 0$ for all $x \in (0,\pi)$.

P4

5/5 a)

$$d_n = \frac{1}{\|\phi_n\|_{L^2}^2}\int_0^\pi f''(x)\,\phi_n(x)\,dx = \frac{-1}{\|\phi_n\|_{L^2}^2}\int_0^\pi f'(x)\phi_n'(x)\,dx +$$

$$+\frac{1}{\|\phi_n\|_{L^2}^2} f'(x)\phi_n(x)\Big]_0^\pi = \frac{1}{\|\phi_n\|_{L^2}^2}\left[\int_0^\pi f(x)\phi_n''(x)\,dx - f(x)\phi_n'(x)\Big]_0^\pi +\right.$$

$$\left. + f'(x)\,\phi_n(x)\Big]_0^\pi\ \right] =$$

$$= \frac{1}{\|\phi_n\|_{L^2}^2}\left(-\lambda_n\int_0^\pi f(x)\phi_n(x)\,dx + \underbrace{\Big[-f(x)\phi_n'(x) + f'(x)\phi_n(x)\Big]_0^\pi}_{=0 \text{ (symmetric B.C.)}}\right) =$$

$$= -\lambda_n c_n \quad //.$$

5/5 b) f'' continuous in $[0,\pi] \Rightarrow \int_0^\pi (f''(x))^2\,dx \leq \left(\max_{x\in[0,\pi]} f''(x)\right)^2 \pi < +\infty$ //. +2

Bessel's inequality:

$$\sum_{n=1}^{\infty} d_n^2\,\|\phi_n\|_{L^2}^2 \leq \|f''\|_{L^2}^2 < +\infty \,. \quad +3$$

(we know it is =, because $\{\phi_n\}_{n\geq 1}$ is complete).

c)

5/5

$$\sum_{n\geq N} |c_n| = \sum_{n\geq N} \frac{1}{|\lambda_n|} |d_n| \leq \qquad \{\text{Schwartz's ineq.}\}$$

$$\leq \left(\sum_{n\geq N} |\lambda_n|^{-2}\right)^{1/2} \left(\sum_{n\geq N} |d_n|^2\right)^{1/2}.$$

d) For large n, $\lambda_n \sim n^2$, so $\sum_{n\geq 1} |\lambda_n|^2 < +\infty$.

5/5

Part b.), $\sum_{n\geq 1} |d_n|^2 < +\infty$.

Therefore, we have

$$\left.\begin{array}{l} \sum_{n\geq 1} c_n \phi_n(x) \text{ with} \\ |c_n \phi_n(x)| \leq c_n \text{ for all } x \in [0,\pi] \\ \qquad\qquad\qquad \text{for all } n \geq N. \\ \sum_{n\geq 1} |c_n| < +\infty \end{array}\right\} \xrightarrow{\text{Weierstrass test}} \begin{array}{l} \sum_{n\geq 1} c_n \phi_n(x) \\ \text{converges uniformly.} \end{array}$$

5/5 e) $\sum_{n\geq 1} c_n \phi_n \to f$ in $L^2(0,\pi)$.

$$\left.\begin{array}{l} \left.\begin{array}{l} \sum_{n\geq 1} c_n \phi_n \text{ converges uniformly,} \\ \phi_n \text{ are continuous} \end{array}\right| \Rightarrow \begin{array}{l} \sum_{n\geq 1} c_n \phi_n \text{ is} \\ \text{continuous in } [0,\pi] \end{array} \end{array}\right\} \Rightarrow$$

$$\Rightarrow \sum_{n\geq 1} c_n \phi_n \to f \quad \text{uniformly in } [0,\pi].$$