MATH 241
LECTURE 15

Higher Dimensional PDEs III
(Bessel Functions) (Section 7.7)

In this lecture we focus on the study of the vibrations of a circular membrane.
To find "explicit" formulas for the solution, we will need to introduce the Bessel functions.

- **Problem:** Solve the wave equation in a circle with Dirichlet BCs

$$\left.\begin{array}{ll} \text{PDE:} & u_{tt} = c^2 \Delta u \quad \text{in } \Omega, \\ \text{BCs:} & u = 0 \quad \text{on } \partial\Omega, \\ \text{ICs:} & u|_{t=0} = f \\ & u_t|_{t=0} = g \end{array}\right\}$$

with Ω a circle of radius R centered at the origin.

- **Remark:** The techniques will be useful to solve many other related problems (damped wave equation, heat equation, etc.) in a circle.

- First, we separate the spatial and time variable. Since we are working in a circle, we choose polar coordinates:

$$u(r,\theta,t) = \phi(r,\theta)\,h(t).$$

- As we did in the previous lecture, we plug this into the PDE to obtain:

$$\phi h'' = c^2 h \,\Delta\phi \Rightarrow \frac{1}{c^2}\frac{h''}{h} = \frac{\Delta\phi}{\phi} = -\lambda \Rightarrow$$

$$\Rightarrow \text{①}\ h''(t) = -\lambda c^2 h(t), \quad \text{②}\ \Delta\phi + \lambda\phi = 0.$$

- BCs: (Dirichlet)

$$u(r,\theta,t) = \phi(r,\theta)h(t) = 0 \text{ on } \partial\Omega \text{ for all } t\geq 0 \Rightarrow$$

$$\Rightarrow \phi \equiv 0 \text{ on } \partial\Omega$$

Notice that we haven't used the fact that we are working with a circle. All this is exactly like the previous day (for general Ω).

$\downarrow$

The problem is reduced to an eigenvalue problem
(two-dimensional)

$$②\quad \left.\begin{array}{l} \Delta\phi + \lambda\phi = 0 \text{ in } \Omega \\ \phi \equiv 0 \text{ on } \partial\Omega \end{array}\right\} \quad \text{and an ODE } ① \; h''(t) = -\lambda c^2 h(t).$$

Obviously, to solve the time ODE we first need to find the eigenvalues from ②.

Nevertheless, we already know that $\lambda > 0$!

So, qualitatively, we can solve ①:

$$h(t) = A_\lambda \cos(\sqrt{\lambda}\, ct) + B_\lambda \sin(\sqrt{\lambda}\, ct).$$

- Let's solve ②. We know all the properties that the eigenfunctions must satisfy (this is the Helmholtz equation with Dirichlet BCs on a smooth domain), so we can easily solve the problem in terms of the eigenfunctions. (do it as an exercise now if it is not in your head right now).

Today, what we want is to actually find the eigenfunctions ϕ for this particular geometry.

• <u>Solving ②</u> : $\left.\begin{array}{ll}\Delta\Phi + \lambda\Phi = 0 & \text{in } \Omega \\ \Phi = 0 & \text{on } \partial\Omega\end{array}\right\}$ <u>Ω a circle</u>.

Ω is a circle with radius R, centered at the origin. Using polar coordinates, the BC reads as

$u(R,\theta) = 0$ for $-\pi \le \theta \le \pi$.

This tells us that separating further r and θ should work:

$u(r,\theta) = F(r)G(\theta)$.

Then,

BC: $u(R,\theta) = F(R)G(\theta) = 0$ for $-\pi \le \theta \le \pi \Rightarrow \boxed{F(R) = 0}$.

We must remember the "hidden" BCs: periodicity in θ, and boundedness at the origen (we assume the membrane does blow up in the middle):

$$G(-\pi) = G(\pi), \qquad |F(0)| < \infty .$$
$$G'(-\pi) = G'(\pi),$$

Recall that, in polar coordinates,

$$\Delta\phi(r,\theta) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}(r,\theta)\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}(r,\theta),$$

so $\Delta\phi + \lambda\phi = 0$ becomes

$$\frac{1}{r}\, G(\theta) \frac{d}{dr}\left(r F'(r)\right) + \frac{1}{r^2} F(r)\, G''(\theta) + \lambda F(r)\, G(\theta) = 0 \rightsquigarrow$$

$$\frac{r}{F(r)} \frac{d}{dr}\left(r F'(r)\right) + \lambda r^2 = - \frac{G''(\theta)}{G(\theta)} = \mu ,$$

where we have introduce a second separation constant (notice that $G''(\theta) + \mu G(\theta) = 0$ gives sines/cosines if μ is positive).

In summary, to solve ② we have now two ODEs :
(1-d eigenvalue problems)

(2.1) $$\left.\begin{aligned} G''(\theta) &= -\mu G(\theta) \\ G(-\pi) &= G(\pi) \\ G'(-\pi) &= G(\pi) \end{aligned}\right\}$$
$(\theta \in [-\pi, \pi])$

(2.2) $$\left.\begin{aligned} &\frac{d}{dr}\left(r F'(r)\right) - \frac{\mu}{r} F(r) + \lambda r F(r) = 0, \\ &F(R) = 0, \qquad r \in [0, R] \\ &|F(0)| < \infty \end{aligned}\right\}$$

- We know that the solution to (2.1) is

$\mu_n = n^2$, $n \geq 0$, with $G_n(\theta) = C_n \sin(n\theta) + D_n \cos(n\theta)$.

In other words, the eigenvalue problem (2.1) generates the "basis" of the classical Fourier Series.

- Then, for each $\mu_n = n^2$, we have to solve (2.2).

Notice that (2.2) is an Sturm-Liouville ODE with

$p(r) = r$, $q(r) = -\frac{n^2}{r}$, $\sigma(r) = r$.

It is not a regular S-L problem:

→ $p(x) = \sigma(x) = r$ are zero at $r=0$ (so not true that $p>0, \sigma>0$ in $[a,b]$).

→ $q(r)$ is not continuous in $[0,R]$ (blows up at $r=0$).

→ $|F(0)|<\infty$ is not Robin, Neumann or Dirichlet.

- Remark: In any case, we know that the Φ do have all the nice properties.
  For $F(r)$, we will study (2.2) in more detail.

- Remark-Exercise: Show that, for each $\mu_n = n^2$, the solutions of (2.2) corresponding to different eigenvalues are orthogonal with weight $\sigma(r) = r$.

Sol: Since this is not a regular S-L problem, we must prove it directly (it is not true "automatically").

Let $F_1(r)$, $F_2(r)$ be two eigenfunctions corresponding to $\lambda_1 \neq \lambda_2$. Then,

$$\frac{d}{dr}\left(r F_1'(r)\right) - \frac{n^2}{r} F_1(r) + \lambda_1 r F_1(r) = 0 \Rightarrow$$

$$\frac{d}{dr}\left(r F_2'(r)\right) - \frac{n^2}{r} F_2(r) + \lambda_2 r F_2(r) = 0 \Rightarrow$$

$$\int_0^R \left(\frac{d}{dr}\left(r F_1'(r)\right)\right) F_2(r)\, dr - n^2 \int_0^R \frac{F_1(r)}{r} F_2(r)\, dr = -\lambda_1 \int_0^R r F_1(r) F_2(r)\, dr,$$

$$\int_0^R \left(\frac{d}{dr}\left(r F_2'(r)\right)\right) F_1(r)\, dr - n^2 \int_0^R \frac{F_2(r)}{r} F_1(r)\, dr = -\lambda_2 \int_0^R r F_2(r) F_1(r)\, dr \Rightarrow$$

$$\Rightarrow (\lambda_2 - \lambda_1) \int_0^R r F_1(r) F_2(r)\, dr = \int_0^R \left(\frac{d}{dr}\left(r F_1'(r)\right)\right) F_2(r)\, dr +$$

$$- \int_0^R \left(\frac{d}{dr}\left(r F_2'(r)\right)\right) F_1(r)\, dr. \qquad (*)$$

Now, we integrate by parts both terms on the right hand side:

$$\int_0^R \frac{d}{dr}\left(r F_1'(r)\right) F_2(r)\, dr = -\int_0^R r F_1'(r) F_2'(r)\, dr + \Big[r F_1'(\cancel{r}) F_2(r) \Big]_0^R$$

$$\int_0^R \frac{d}{dr}\left(r F_2'(r)\right) F_1(r)\, dr = -\int_0^R r F_2'(r) F_1'(r)\, dr + \Big[r F_2'(\cancel{r}) F_1(r) \Big]_0^R$$

The boundary terms vanish because $F_2(R)=0, F_1(R)=0$.

Therefore, going back to (*),

$$\underbrace{(\lambda_2-\lambda_1)}_{\neq 0}\int_0^R r F_1(r) F_2(r)\, dr = 0 \rightarrow \int_0^R r F_1(r) F_2(r)\, dr = 0 \; //.$$

- Remark: This method is indeed the same that proves the orthogonality property for regular S-L problems.
  ↳ It is basically an application of integration by parts.

- Solving (2.2):

$$\begin{cases} \frac{d}{dr}\left(r F'(r)\right) - \frac{n^2}{r} F(r) + \lambda r F(r) = 0, & 0<r<R, \\ F(R)=0 & (n\geq 0) \\ |F(0)|<\infty. & \end{cases}$$

The difficulty here comes from solving the non-constant coefficients second-order ODE.

We dedicate a new section to analyse this ODE.

Once we understand the solutions of the ODE, we will be able to solve (2.2), and similar problems with different BCs.

Bessel Functions

We wanted to solve the ODE

$$\frac{d}{dr}(r F'(r)) - \frac{n^2}{r} F(r) + \lambda r F(r) = 0 \, , \text{ for each } n = 0, 1, 2, \dots$$

We first rewrite it as

$$r^2 F''(r) + r F'(r) + (\lambda r^2 - n^2) F(r) = 0 \, ,$$

so that it reminds us to an Euler ODE (but that is the case only when $\lambda = 0$).

→ A nice trick allows us to get rid of λ. Make the change of variable given by

$\boxed{z = \sqrt{\lambda}\, r}$ (which is well defined for $\underline{\lambda > 0}$)

[if we had $\lambda = 0$, we solve that case apart: it is an Euler ODE! ⟿ we know this for our problem.]

After changing variables, the ODE becomes

$$z^2 F''(z) + z F'(z) + (z^2 - n^2) F(z) = 0 \, .$$

[→ You can go directly to page -173- and read later about Bessel functions].

Change of variables (and chain rule)

Briefly: $$\left.\begin{array}{l}\dfrac{dF}{dr} = \dfrac{dF}{dz}\dfrac{dz}{dr} \longrightarrow \dfrac{d^2F}{dr^2} = \dfrac{d^2F}{dz^2}\left(\dfrac{dz}{dr}\right)^2 + \dfrac{dF}{dz}\dfrac{d^2z}{dr^2} \\[2ex] \dfrac{dz}{dz} = \sqrt{\lambda}, \quad \dfrac{d^2z}{dr^2} = 0 \end{array}\right\} \rightarrow$$

$$\rightarrow r^2 F''(r) \rightsquigarrow \frac{z^2}{\cancel{\lambda}}\cancel{\lambda} F''(z)$$

$$r F'(r) \rightsquigarrow \frac{z}{\cancel{\sqrt{\lambda}}}\cancel{\sqrt{\lambda}} F'(z)$$

$$(\lambda r^2 - n^2) F(r) \rightsquigarrow \left(\cancel{\lambda}\frac{z^2}{\cancel{\lambda}} - n^2\right) F(z).$$

(not needed)

↓

• More precisely, we are defining a new function:

$$\left.\begin{array}{l}\tilde{F} : \mathbb{R} \longrightarrow \mathbb{R} \\ \quad z \longmapsto F(z)\end{array}\right\} \quad \text{such that} \quad F(r) = \tilde{F}(z(r)).$$

That is, if we denote the change of variables as a function z:

$$\left.\begin{array}{l}z : \mathbb{R} \longrightarrow \mathbb{R} \\ \quad r \longmapsto z(r) = \sqrt{\lambda}\, r\end{array}\right\},$$

we are defining $\tilde{F}$ by $F = \tilde{F} \circ z$ (composition).

Then, the chain rule says that

$$\frac{dF}{dr}(r) = \frac{d}{dr}\left(\tilde{F}(z(r))\right) = \tilde{F}'(z(r))\, z'(r) = \tilde{F}'(z(r))\sqrt{\lambda},$$

$$\frac{d^2F}{dr^2}(r) = \frac{d}{dr}\frac{dF}{dr}(r) = \frac{d}{dr}\left(\tilde{F}'(z(r))\sqrt{\lambda}\right) =$$

$$= \tilde{F}''(z(r))\, z'(r)\sqrt{\lambda}.$$

That is, $\frac{dF}{dr}(r) = \tilde{F}'(z(r))\sqrt{\lambda}$, $\frac{d^2F}{dr^2}(r) = \tilde{F}''(z(r))\,\lambda$.

Going into the ODE, with $r = \frac{z}{\sqrt{\lambda}}$,

$$\frac{(z(r))^2}{\cancel{\lambda}}\tilde{F}''(z(r))\cancel{\lambda} + \frac{z(r)}{\cancel{\sqrt{\lambda}}}\tilde{F}'(z(r))\cancel{\sqrt{\lambda}} + \left(\cancel{\lambda}\frac{(z(r))^2}{\cancel{\lambda}} - n^2\right)\tilde{F}(z(r)) = 0,$$

Since the change of variables $z = \sqrt{\lambda}\, r$ is a bijection (i.e., one-to-one), we can simply write

$$z^2\tilde{F}''(z) + z\tilde{F}'(z) + (z^2 - n^2)\tilde{F}(z) = 0.$$

In practice, people don't usually make a distinction in notation for F and $\tilde{F}$

(we simply write $F(r)$ and $F(z)$, understanding implicitly the composition that gives the change of variables).

The ODE $\boxed{z^2 F''(z) + z F'(z) + (z^2 - n^2) F(z) = 0}$ is called Bessel's differential equation of order n.

Remark: If we solve Bessel's diff. equation, then we have solved our initial ODE by changing back $z = \sqrt{\lambda}\, r$.

- **Theorem**: The general solution of a second-order linear and homogeneous ODE is a linear combination of two independent solutions.

(Proof: linear algebra based $\rightsquigarrow$ ODE course).

We've seen many examples of this fact before:

$y'' + y = 0 \rightsquigarrow y(x) = c_1 \cos(x) + c_2 \sin(x)$

$y'' - y = 0 \rightsquigarrow y(x) = c_1 \cosh(x) + c_2 \sinh(x)$

$\vdots$

$\rightarrow$ Notice that Bessel's ODE of order n is linear and homogeneous.

• Claim: The general solution to Bessel's ODE of order n, $z^2 F''(z) + zF'(z) + (z^2 - n^2)F(z) = 0$, is given by

$$F(z) = c_1 J_n(z) + c_2 Y_n(z).$$

where $J_n(z) \equiv$ Bessel function of the first kind,
$Y_n(z) \equiv$ Bessel function of the second kind.

The functions $J_n(z)$ and $Y_n(z)$ cannot be written in terms of the typical elementary functions. We will show later that they can be defined as infinite series:

[see page].

$$J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k+n}}{k!(k+n)!},$$

$$Y_n(z) = \frac{2}{\pi}\left[\left(\log\left(\frac{z}{2}\right) + \gamma\right) J_n(z) - \frac{1}{2}\sum_{k=0}^{n-1} \frac{(n-k-1)!\,(z/2)^{2k-n}}{k!} + \right.$$

$$\left. + \frac{1}{2}\sum_{k=0}^{\infty} (-1)^{k+1}\big(g(k) + g(k+n)\big)\frac{(z/2)^{2k+n}}{k!(n+k)!}\right], \text{ where}$$

$g(k) = \sum_{l=0}^{k} \frac{1}{l}$ with $g(0) = 0$,

$\gamma = \lim_{k\to\infty} (g(k) - \log(k)) \simeq 0.577$ (Euler's constant)

(and the convection that $\sum_{k=0}^{-1} (..) \equiv 0$).

The above formulas don't tell us much at first. So let's get more intuition about these new functions.

Singular Points of ODEs

Given a second-order linear ODE $F''(z) + a(z)F'(z) + b(z)F(z) = 0$, we call $z = z_0$

- an ordinary point if $a(z), b(z)$ and all their derivatives are finite at $z = z_0$.
- a singular point if it is not an ordinary point.

Bessel ODE has $a(z) = \frac{1}{z}$ and $b(z) = 1 - \frac{n^2}{z^2}$. Thus the only singular point is $z = 0$.

Claim: (ODE theory) The solutions of an ODE are well-behaved in a neighborhood of an ordinary point. (that is, the solution and its derivatives exist).

→ So $J_n(z), Y_n(z)$ are smooth functions outside of $z = 0$.

We will first understand what happens at $z=0$. Then we will give some intuition of the global behaviour of $J_n(z)$, $Y_n(z)$

1) Small z ($z \approx 0$)

We can use the formulas from page -168-.
Notice that if z is very small, then bigger powers can be neglected ($z^2 << z$ for $z \approx 0$; for example, $(0.01)^2 = 0.0001$).

So we can see that:

$$J_n(z) = \frac{1}{n!}\left(\frac{z}{2}\right)^n - \underbrace{\frac{1}{(n+1)!}\left(\frac{z}{2}\right)^n\left(\frac{z}{2}\right)^2 + \dots}_{\text{much smaller}} \approx \frac{1}{2^n n!} z^n \ , \ n \geq 0,$$

For $Y_n(z)$ we do the cases $n=0$ and $n \geq 1$ separately:

$\underline{n=0}$

$$Y_0(z) = \frac{2}{\pi}\left(\log\left(\frac{z}{2}\right) + \gamma\right) J_0(z) + \frac{1}{2}\sum_{k=0}^{\infty} (-1)^{k+1} 2y(k) \frac{(z/2)^{2k}}{k!\,k!} \approx$$

$$\approx \frac{2}{\pi}\log\left(\frac{z}{2}\right) + \frac{2}{\pi}\gamma - \frac{1}{2}2\overbrace{y(0)}^{=0} + \frac{1}{2}2y(1)\left(\frac{z}{2}\right)^2 + \dots \approx$$

$$\approx \frac{2}{\pi}\log\left(\frac{z}{2}\right) \qquad \left(\text{notice that near } z=0 \quad \log(z) \gg \gamma\right)$$

<u>$n \geq 1$</u>

$$Y_n(z) \approx \frac{2}{\pi}\left(\log\left(\frac{z}{2}\right)+\gamma\right)\frac{z^n}{2^n n!} - \frac{1}{\pi}\sum_{k=0}^{n-1}\frac{(n-k-1)!\,(z/2)^{2k-n}}{k!} +$$

$$+\frac{1}{\pi}\sum_{k=0}^{\infty}(-1)^{k+1}\left(\psi(k)+\psi(k+n)\right)\frac{(z/2)^{2k+n}}{k!\,(n+k)!} \approx$$

$$\approx \frac{2}{\pi}\log\left(\frac{z}{2}\right)\frac{z^n}{2^n n!} - \frac{1}{\pi}(n-1)!\,\frac{z^{-n}}{2^{-n}} \approx -\frac{2^n(n-1)!}{\pi}\,\frac{1}{z^n}.$$

In summary, the behaviour of $J_n(z)$, $Y_n(z)$ near $z=0$ is given by:

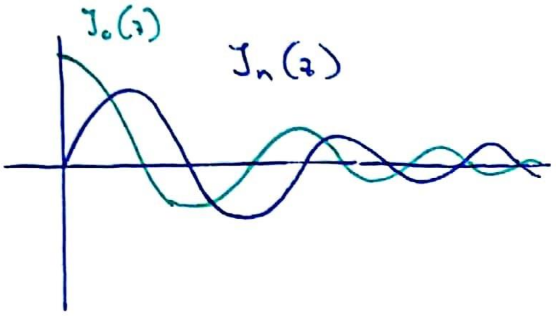
$$\left.\begin{aligned} J_n(z) &\approx \frac{1}{2^n n!}z^n, \\ Y_n(z) &\approx \begin{cases} \frac{2}{\pi}\log\left(\frac{z}{2}\right) & \text{if } n=0, \\ -\frac{2^n(n-1)!}{\pi}\,\frac{1}{z^n} & \text{if } n\geq 1. \end{cases} \end{aligned}\right\} \text{ for } z\approx 0.$$

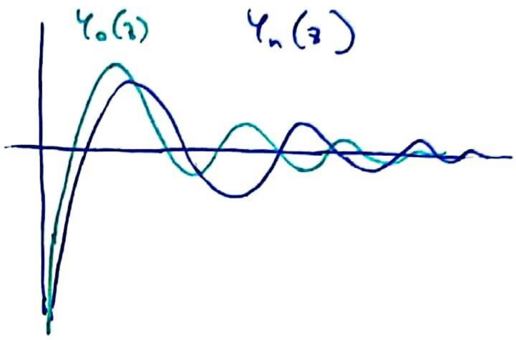
• <u>Remark</u>:

$$\lim_{z\to 0} J_n(z) = \begin{cases} 1 & \text{if } n=0, \\ 0 & \text{if } n\geq 1. \end{cases}$$

$$\lim_{z\to 0} Y_n(z) = \infty \quad \text{for all } n\geq 0.$$

We know now how Bessel functions look like near $z=0$. Similar asymptotic formulas for large z are much harder to obtain. For our purposes, we just need to know the rough shape:





The important point here is:

Bessel functions look like decaying oscillations. Therefore, for fixed n, both $Y_n(z)$ and $J_n(z)$ have infinitely many roots.

<u>Note</u>: (optional) If you'd like some intuition about why Bessel functions have these shapes, you can read section 7.8.1.

- We now go back to our wave equation problem.

[Continuation of pages -163-, -164-]

Solving (E.2) : $\frac{d}{dr}\left(r F'(r)\right) - \frac{n^2}{r} F(r) + \lambda r F(r) = 0$, $0 < r < R$,

$F(R) = 0$, $(n \geq 0)$

$|F(0)| < \infty$,

We changed variables $z = \sqrt{\lambda}\, r$ to transform the ODE into the Bessel ODE

$z^2 F''(z) + z F'(z) + (z^2 - n^2) F(z) = 0$.

We claimed that the general solution is

$F(z) = c_1 J_n(z) + c_2 Y_n(z)$, where $J_n(z), Y_n(z)$ are the Bessel functions of 1^{st} and 2^{nd} kind. These are new functions whose shape is given on page -172-

(in practice, the graphs can be used as "definitions", similarly to the way we think about e^x, $\sin(x)$, $\sqrt{x}$, ...).

We know their approximations for small z.

We know that J_n, Y_n is a family of orthogonal functions with weight r (page -161-, -162-).

↑ summary of Bessel functions.

Going back to r:

$$F(r) = c_1 J_n(\sqrt{\lambda} r) + c_2 Y_n(\sqrt{\lambda} r).$$

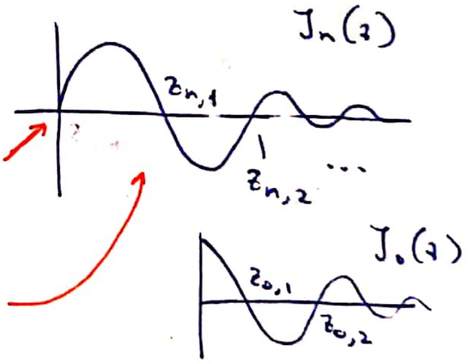
We impose the BCs:

· $|F(0)| < \infty \Rightarrow c_2 = 0$ (because Y_n blows up at 0).

Then,

$$F(R) = 0 = c_1 J_n(\sqrt{\lambda} R),$$

which determines the values of λ. If we denote $z_{n,m}$ the mth root of $J_n(z)$ (because we don't know their values, as opposed to what happened with $\sin(\sqrt{\lambda} L) = 0 \ldots$), then



$$J_n(\sqrt{\lambda} R) = 0 \Leftrightarrow$$

not possible

$$\Leftrightarrow \sqrt{\lambda} R = z_{n,m} .$$ (we know $\lambda > 0$)

That is, for each n, we have infinitely many eigenvalues. We should better write

$$\lambda_{n,m} = \left(\frac{z_{n,m}}{R}\right)^2 \qquad (n \geq 0, m \geq 1)$$

from page -160-

we started counting the roots of $J_n(z)$ at $m = 1$.

In summary, the solution to (2.2) is

$$\lambda_{n,m} = \left(\frac{z_{n,m}}{R}\right)^2 \quad , \text{ with } F_{n,m}(r) = J_n\left(\sqrt{\lambda_{n,m}}\, r\right) = J_n\left(z_{n,m}\frac{r}{R}\right).$$

- Joining everything together, we had that

$u(r,\theta,t) = \phi(r,\theta)\, h(t)$, with

(1) $h''(t) = -\lambda c^2 h(t)$ (2) $\Delta\phi + \lambda\phi = 0$ in Ω

$\phi = 0$ on $\partial\Omega$ $\rightarrow \phi(r,\theta) = F(r)G(\theta)$

(2.1) $G''(\theta) = -\mu G(\theta)$

$G(-\pi) = G(\pi)$

$G'(-\pi) = G'(\pi)$

(2.2) $\frac{d}{dr}\left(rF'(r)\right) - \frac{n^2}{r}F(r) + \lambda r F(r) = 0$

$F(R) = 0$

$|F(0)| < \infty$.

with solutions given by:

(2.1) $\rightarrow G_n(\theta) = C_n \sin(n\theta) + D_n \cos(n\theta)$, $\mu_n = n^2$, $n \geq 0$,

(2.2) $\rightarrow F_{n,m}(r) = J_n\left(z_{n,m}\frac{r}{R}\right)$, $\lambda_{n,m} = \left(\frac{z_{n,m}}{R}\right)^2$. $n \geq 0$, $m \geq 1$

(1) $\rightarrow h(t) = A_\lambda \cos(\sqrt{\lambda}ct) + B_\lambda \sin(\sqrt{\lambda}ct)$.

Before applying superposition, I prefer to combine (2.1) and (2.2) to have the complete solution to (2).

$$② \rightarrow \Phi_{n,m}(r,\theta) = J_n\left(z_{n,m}\frac{r}{R}\right)\left(C_n \sin(n\theta) + D_n \cos(n\theta)\right).$$

Finally, the solution in u is:

$$u(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} h_{n,m}(t)\,\Phi_{n,m}(r,\theta) =$$

$$= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n\left(z_{n,m}\frac{r}{R}\right)\left(C_n \sin(n\theta) + D_n\cos(n\theta)\right)$$

$$\cdot\left(A_{n,m}\cos\left(\frac{z_{n,m}}{R}ct\right) + B_{n,m}\sin\left(\frac{z_{n,m}}{R}ct\right)\right).$$

• Initial conditions:

$$u(r,\theta,0) = f(r,\theta) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n\left(z_{n,m}\frac{r}{R}\right)A_{n,m}\left(C_n\sin(n\theta) + D_n\cos(n\theta)\right) =$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=1}^{\infty} J_n\left(z_{n,m}\frac{r}{R}\right)A_{n,m}C_n\right)\sin(n\theta) +$$

$$+ \sum_{n=0}^{\infty}\left(\sum_{m=1}^{\infty} J_n\left(z_{n,m}\frac{r}{R}\right)A_{n,m}D_n\right)\cos(n\theta) \rightarrow$$ (classical Fourier series)

$$\sum_{m=1}^{\infty} J_n\left(z_{n,m}\frac{r}{R}\right)A_{n,m}C_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(r,\theta)\sin(n\theta)\,d\theta$$

$$\sum_{m=1}^{\infty} J_n\left(z_{n,m}\frac{r}{R}\right)A_{n,m}D_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(r,\theta)\cos(n\theta)\,d\theta \quad n\neq 0$$

$$\sum_{m=1}^{\infty} J_0\left(z_{0,m}\frac{r}{R}\right)A_{0,m}D_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(r,\theta)\,d\theta.$$

Now we should use orthogonality of J_n in r.

And then repeat for $u_t(r,\theta,0)$, obtaining four equations for $C_n, D_n, A_{n,m}, D_{n,m}$.

→ It is faster to use the 2-d orthogonality of the eigenfunctions in ②:

$$u(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} h_{n,m}(t)\left(C_n \Phi^s_{n,m}(r,\theta) + D_n \Phi^c_{n,m}(r,\theta)\right),$$

where Φ^s denotes the part with $\sin(n\theta)$, and Φ^c with $\cos(n\theta)$.

Then,

$$f(r,\theta) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \underbrace{h_{n,m}(0)}_{=A_{n,m}}\left(C_n \Phi^s_{n,m}(r,\theta) + D_n \Phi^c_{n,m}(r,\theta)\right) \Rightarrow$$

$$\Rightarrow \int_\Omega f \Phi^s_{k,\ell}\, dA = A_{k,\ell} C_k \int_\Omega (\Phi^s_{k,\ell})^2 \, dA \Rightarrow A_{k,\ell} C_k = \frac{\int_\Omega f \Phi^s_{k,\ell}\, dA}{\int_\Omega (\Phi^s_{k,\ell})^2\, dA},$$

and same with $\Phi^c_{n,m}$:

$$A_{k,\ell} D_k = \frac{\int_\Omega f \Phi^c_{k,\ell}\, dA}{\int_\Omega (\Phi^c_{k,\ell})^2\, dA}.$$

Do the same for $B_{n,m}$: $\quad h'_{n,m}(0) = B_{n,m} \frac{z_{n,m}}{R} c.$

$$g(r,\theta) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} h'_{n,m}(0)\left(C_n \Phi^s_{n,m}(r,\theta) + D_n \Phi^c_{n,m}(r,\theta)\right) //.$$