MATH 425
LECTURE 17 : Harmonic Functions.

In this lecture we focus on some properties that harmonic functions satisfy.

1.) Rotational Invariance

Harmonic functions don't need to be rotationally symmetric. But the Laplacian operator is rotational invariant, meaning that its expression doesn't change if we rotate the coordinates. For this reason the Laplacian is used as a model for isotropic diffusion (that is, diffusion that acts equally in all direction).

Let's check it: (in 2d for simplicity)

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{array}{l} \tilde{x} = x\cos\alpha + y\sin\alpha, \\ \tilde{y} = -x\sin\alpha + y\cos\alpha, \end{array}$$

Thus,

$$u_x = u_{\tilde{x}}\cos\alpha - u_{\tilde{y}}\sin\alpha,$$

$$u_y = u_{\tilde{x}}\sin\alpha + u_{\tilde{y}}\cos\alpha,$$

$$u_{xx} = (u_{\tilde{x}}\cos\alpha - u_{\tilde{y}}\sin\alpha)_{\tilde{x}}\cos\alpha - (u_{\tilde{x}}\cos\alpha - u_{\tilde{y}}\sin\alpha)_{\tilde{y}}\sin\alpha =$$

$$= u_{\tilde{x}\tilde{x}}\cos^2\alpha - u_{\tilde{y}\tilde{x}}\sin\alpha\cos\alpha - u_{\tilde{x}\tilde{y}}\cos\alpha\sin\alpha + u_{\tilde{y}\tilde{y}}\sin^2\alpha,$$

$$u_{yy} = \dots = u_{\tilde{x}\tilde{x}}\sin^2\alpha + 2u_{\tilde{x}\tilde{y}}\sin\alpha\cos\alpha + u_{\tilde{y}\tilde{y}}\cos^2\alpha,$$

so

$$\Delta u = u_{xx} + u_{yy} = u_{\tilde{x}\tilde{x}} + u_{\tilde{y}\tilde{y}} = \tilde{\Delta} u.$$

- This invariance indicates that there might exist harmonic functions that are also rotationally invariant. That is, functions $u(r,\theta)$ with $\Delta u = 0$ that only depend on r.

It is easy to find these functions (both in 2d, 3d), and they are important if you proceed to study Green's functions (very important in physics).

2.1 Maximum Principle

Statement: Let $D \subseteq \mathbb{R}^2$ be an open connected bounded set. Let $u(x,y)$ be a harmonic function, continuous on $\bar{D} = D \cup \partial D$.

Then,

(notation for (x,y))

1) [Weak max. principle] $\max_{\vec{x} \in \bar{D}} u(\vec{x}) = \max_{\vec{x} \in \partial D} u(\vec{x})$,

2) [Strong max. principle], $u(\vec{x}) < \max_{\vec{x} \in \partial D} u(\vec{x})$ for all $\vec{x} \in \bar{D}$.

(unless u is constant)

- The proof for the weak max. principle is like the one for the heat equation. Let's do it as an exercise.

Proof: [Weak]

Define the auxiliary function $v(\vec{x}) = u(\vec{x}) + \varepsilon |\vec{x}|^2$, which will allow us to get a contradiction.

Remark: Remember that the idea was: suppose u attain its maximum inside D, then $\Delta u \leq 0$. This is "almost" a contradiction $\to$ we'd need a strict $<$. The term $\varepsilon |\vec{x}|^2$ will help.

- For v, $\Delta v = \Delta u + \varepsilon \Delta |\vec{x}|^2 = \underbrace{\Delta u}_{\substack{= 0 \\ (u \text{ is harmonic})}} + 4\varepsilon = 4\varepsilon > 0$ in D.

Therefore, v cannot attain a maximum in the interior of D (otherwise $\Delta v \leq 0$ at that point $\nleftarrow$).

- Since v is continuous on $\bar{D}$, and $\bar{D}$ is closed and bounded, v has to attain its maximum somewhere in $\bar{D}$, so therefore in ∂D. Denote such a point by $\vec{x}^* \in \partial D$.

Then,

$$u(\vec{x}) \leq v(\vec{x}) \leq v(\vec{x}^*) = u(\vec{x}^*) + \varepsilon |\vec{x}^*|^2 \leq$$

$$\leq \max_{\vec{x} \in \partial D} u(\vec{x}) + \varepsilon L^2,$$

where $L = \max_{\vec{x} \in \partial D} |\vec{x}|$ [maximum distance from ∂D to the origin]

• Because $\varepsilon > 0$ was arbitrary,

$$u(\vec{x}) \leq \max_{\vec{x} \in \partial D} u(\vec{x}) + \varepsilon L^2 \xrightarrow{(\varepsilon \to 0^+)} \underset{\underset{\text{any } \vec{x} \in \bar{D}}{\uparrow}}{u(\vec{x})} \leq \max_{\vec{x} \in \partial D} u(\vec{x}) \quad //.$$

$\square$.

• To prove the strong version we need some more properties about harmonic functions. First, let's state properly what we discovered in last lecture:

• Theorem: (Poisson's formula)

Let $h(\theta)$ be a continuous function, $\theta \in [0, 2\pi)$. Then, using the notation $w(\vec{x}') = h(\theta)$, it holds that

1) $$u(\vec{x}) = \frac{L^2 - |\vec{x}|^2}{2\pi} \int_{|\vec{x}'| = L} \frac{w(\vec{x}')}{|\vec{x} - \vec{x}'|^2} \frac{ds}{L}$$ is harmonic in D (circle radius L),

2) It is the only harmonic function in D such that

$$\lim_{\vec{x} \to \vec{x}_0} u(\vec{x}) = w(\vec{x}_0) \text{ for all } \vec{x}_0 \in \partial D.$$

(so we can write $u(\vec{x}')$ in Poisson's formula)

• That is, the solution we obtain as a candidate by using the method of separation of variables is an actual solution (harmonic, continuous up to the boundary, and satisfies the BC pointwise).

• Proof: Book (not required, recommended for math majors).

• Proposition: Mean Value Property

Let u be harmonic in a disk D, continuous in $\bar{D}$, $\Big\}$ then,

The value of u at the center of D $=$ Average of u along ∂D.

Proof: We can assume the center is $\vec{x}_c = \vec{0}$ (origin) without loss of generality (make a translation otherwise; Δ is invariant under translations).

Poisson's formula evaluated at $\vec{x} = \vec{0}$ yields the result:

$$u(\vec{0}) = \frac{L^2 - |\vec{0}|^2}{2\pi} \int_{|\vec{x}'|=L} \frac{u(\vec{x}')}{|\vec{0} - \vec{x}'|^2} \frac{ds}{L} = \frac{1}{2\pi L} \int_{|\vec{x}'|=L} u(\vec{x}')\, ds$$

$|\vec{x}'|^2 = L^2$

• Proof of <u>Strong Max. Principle</u>: [consider $u \not\equiv$ constant]

By the weak max. principle, we know the maximum of u is attained at some point $\vec{x}^* \in \partial D$.

We want to prove it is not attained at any point $\vec{x} \in D$.

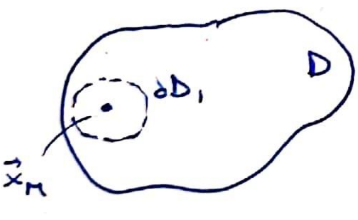
Assume the contrary, i.e., assume that

$$\exists\, \vec{x}_M \in D \text{ such that } u(\vec{x}_M) = u(\vec{x}^*).$$

Then,

$$u(\vec{x}) \leq u(\vec{x}_M) = M \quad \forall \vec{x} \in D.$$

Now, since $\vec{x}_M \in D$ and D is open, we can draw a circle around $\vec{x}_M$ completely contained inside D:



By the Mean Value Property, $u(\vec{x}_M)$ has to be equal to the average of u on that circle. But, since $u(\vec{x}_M)$ is the maximum value, $u(\vec{x}) \leq u(\vec{x}_M)$ for any $\vec{x}$ on that circle ∂D_1.

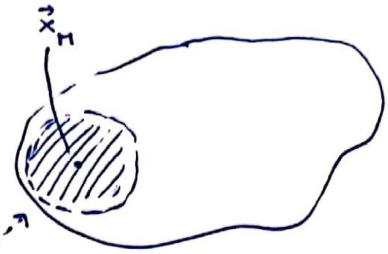
The only way then that the average on ∂D_1 equals $u(\vec{x}_M)$ is that

$$u(\vec{x}) = M \text{ for all } \vec{x} \in \partial D_1.$$

But this is true for all the other circles centered at $\vec{x}_M$ and contained in D,

so

$u(\vec{x}) = M$ for all $\vec{x}$ in that disk



Now pick any of those $\vec{x}$, and repeat the argument with this new point $\vec{x}$ as the center. You will a new disk where all the points satisfy that $u(\vec{x}) = M$. Because D is connected, we can reach all points $\vec{x} \in D$ by this process, concluding that $u(\vec{x}) = M \ \forall \vec{x} \in D$, that is, u is constant $\rightarrow\!\!\!\!\leftarrow$ contradiction,

thus $\nexists \ \vec{x}_M \in D \ / \ u(\vec{x}_M) = u(\vec{x}^*)$ .//.

• Proposition: Smoothness of harmonic functions.

If u is harmonic in an open set $\Omega \subseteq \mathbb{R}^2$, then

$$u \in \underbrace{C^\infty(D)}_{\substack{\text{all partial derivatives of any order} \\ \text{exist and are continuous.}}}$$

Proof:

Consider any point $\vec{x} \in D$. Since D is open, consider a circle centered at $\vec{x}$ and contained in D (with radius say $\varepsilon > 0$).

Then

$$u(\vec{x}) = \frac{\varepsilon^2 - |\vec{x}|^2}{2\pi} \int_{|\vec{x}| = \varepsilon} \frac{u(\vec{x}')}{|\vec{x} - \vec{x}'|^2} \frac{ds}{\varepsilon}.$$

Now, the derivatives will hit the term $\varepsilon^2 - |\vec{x}|^2$ or the denominator inside the integral, $\frac{1}{|\vec{x} - \vec{x}'|^2}$. Both terms are infinitely differentiable (the second one because $\vec{x} \notin \partial D$ and $\vec{x}' \in \partial D$).

This was true for any point in D, thus $u \in C^\infty(D)$ //.