MATH 425
LECTURE 20 : Poisson's Equation

The Poisson equation is the nonhomogeneous version of the Laplace equation, which can be interpreted as the time-independent version of the heat equation.

- We will see how to use the method of eigenfunctions expansion. Since there is no time variable, we won't obtain an ODE as in lecture 19.

Problem:

PDE: $\Delta u = Q$ in Ω

BC: $u = f$ on $\partial\Omega$

Ω some domain in $\mathbb{R}^2$ (or $\mathbb{R}^3$).

[Q, f are given functions]

we could consider different BCs in different parts of the boundary.

Solution:

- First, consider the associated modified homogeneous problem:

$$\begin{cases} \Delta\phi = -\lambda\phi & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega \end{cases} \quad \text{(Helmholtz equation)}$$

same type of BC as in the original problem.

This eigenvalue problem will have solutions λ_i, ϕ_i.

Remark: For Ω a rectangle, circle, cylinder, sphere (or modified versions of them) we have already solved the Helmholtz equation with given BCs, having explicit λ_i, ϕ_i.

- Once we have λ_i, ϕ_i, we let our solution be an expansion in these eigenfunctions:

  $u = \sum_i c_i \phi_i$, for some constants c_i to be determined.

Remark: We might be tempted to say here: "Since $u = \sum_i c_i \phi_i$, we substitute into the PDE to find that $\Delta u = \cancel{\sum_i} c_i \Delta \phi_i = Q \Rightarrow -\sum_i c_i \lambda_i \phi_i = Q \Rightarrow$

orthogonality $\Rightarrow$ $-c_n \lambda_n \int_\Omega (\phi_n)^2 = \int_\Omega \phi_n Q \Rightarrow c_n = -\frac{1}{\lambda_n} \frac{\int_\Omega \phi_n Q}{\int_\Omega (\phi_n)^2}$.

$\hookrightarrow$ The idea is "almost" right, but notice that we haven't used the BCs at all! Something must be wrong...

→ We cannot differentiate term by term on

$u = \sum_i c_i \phi_i$, because u and ϕ_i do not

satisfy the same BCs.

[In Lect. 19 we assumed the BCs for u were homogeneous, because in Lect. 18 we showed that we could always make the BCs homogeneous by changing the force].

• So far we have that

① $u = \sum_i c_i \phi_i$ with $\left.\begin{aligned} \Delta\phi_i &= -\lambda_i \phi_i \quad \text{in } \Omega \\ \phi_i &= 0 \quad \text{on } \partial\Omega \end{aligned}\right\}$ ②

We can use orthogonality as usual to find c_i in terms of u (but now this is our unknown) and ϕ_i:

$$u = \sum c_i \phi_i \Rightarrow c_i = \frac{\int_\Omega u \phi_i}{\int_\Omega (\phi_i)^2}.$$

We use now ② to substitute on the numerator

$\phi_i = \frac{-1}{\lambda_i} \Delta \phi_i$, so that

$$c_i = \frac{-1}{\lambda_i \int_\Omega (\phi_i)^2} \int_\Omega u \, \Delta \phi_i ,$$

and now we integrate by parts twice,

$$c_i = \frac{-1}{\lambda_i \int_\Omega (\phi_i)^2} \Big(\int_\Omega \phi_i \underbrace{\Delta u}_{= Q} - \underbrace{\cancel{\int_{\partial\Omega} \phi_i \, \vec{n} \cdot \nabla u}}_{=0} + \int_{\partial\Omega} \underbrace{u}_{= f} \, n \cdot \nabla \phi_i \Big) .$$

so in conclusion

this is the boundary term we were missing on page -224- .

$$\boxed{c_i = -\frac{1}{\lambda_i} \frac{\int_\Omega \phi_i Q + \overbrace{\int_{\partial\Omega} f \, n \cdot \nabla \phi_i}}{\int_\Omega (\phi_i)^2} , \qquad u = \sum_i c_i \phi_i}$$

the formula will be different for different BC

Summary:

$$\left.\begin{array}{ll} \Delta u = Q & \text{in } \Omega \\ \text{BCs} \ldots & \partial\Omega \end{array}\right\}$$

(same type)

1) $u = \sum_i c_i \phi_i$ with $\left.\begin{array}{ll} \Delta \phi_i = -\lambda_i \phi_i & \Omega \\ \text{homogeneous BCs} \ldots & \partial\Omega \end{array}\right\}$

2) $$c_i = \frac{\int_\Omega u\, \phi_i}{\int_\Omega (\phi_i)^2} = \frac{-1}{\lambda_i} \frac{\int_\Omega u\, \Delta\phi_i}{\int_\Omega (\phi_i)^2} = \ldots$$ integrate by parts, use BCs, and substitute $\Delta u = Q$. //.

Remark: A second option to solve these problems is to split them into two. For example,

$$\left.\begin{array}{ll} \Delta u = Q & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{array}\right\} \longrightarrow \left.\begin{array}{ll} \Delta u_1 = 0 & \text{in } \Omega \\ u_1 = f & \text{on } \partial\Omega \end{array}\right\} \quad \left.\begin{array}{ll} \Delta u_2 = Q & \text{in } \Omega \\ u_2 = 0 & \text{on } \partial\Omega \end{array}\right\}$$

with $u = u_1 + u_2$.

- Notice that from previous lectures we know how to solve for u_1.
- For u_2, because the BCs are already homogeneous, we can differentiate term by term in $u = \sum_i c_i \phi_i$!