Name ______________________________

Math 425 - Final Exam - Part II
April 30 - May 7, 2020.

You are expected to uphold the Code of Academic Integrity. I certify that all of the work on this test is my own.

Signature: ______________________

The exam is open book. Correct answers without proper justification will not receive full credit. Clearly highlight your answers and the steps taken to arrive at them: illegible work will not be graded. **Solve each problem in a separate page, and always indicate the number and part of the problem you are solving.** When you finish, scan all your work and **upload a PDF to Canvas** (you may use CamScanner app). Please **do not upload photos**.

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Problem	Points	Your score
1	30	
2	30	
3	30	
Total	90	
Standardized Total	$\frac{100}{90}\cdot$Total	

Problem 1 [30 points]

Part a. [20 points] Consider the inhomogeneous heat equation with periodic boundary conditions on the interval $(-\pi, \pi)$:

$$\begin{aligned}
&u_t(x,t) = u_{xx}(x,t) + Q(x,t), \quad -\pi < x < \pi, \quad t > 0,\\
&u(-\pi,t) = u(\pi,t), \quad u_x(-\pi,t) = u_x(\pi,t),\\
&u(x,0) = u_0(x).
\end{aligned}$$

Assume u_0 and Q are smooth enough. Consider the eigenfunction expansion for the solution

$$u(x,t) = \sum_{n\geq 0} a_n(t)\cos(nx) + \sum_{n\geq 1} b_n(t)\sin(nx).$$

a.1) [10 points] Find the ordinary differential equations that the coefficients a_n and b_n must satisfy.

a.2) [10 points] Find the formulas for a_n and b_n in terms of the Fourier coefficients of u_0 and Q.

Part b. [10 points] Consider the inhomogeneous heat equation in the whole line

$$\begin{aligned}
&u_t(x,t) = u_{xx}(x,t) + Q(x,t), \quad -\infty < x < \infty, \quad t > 0,\\
&u(x,0) = u_0(x),
\end{aligned}$$

with the condition that solutions must decay to zero at infinity. Assume that u_0 and Q have well-defined Fourier Transforms.

b.1) [5 points] Find the Fourier Transform of the solution u in terms of the Fourier Transform of u_0 and Q. Notice the similarity between your solution and the formulas in Part a.2.

b.2) [5 points] Invert Part b.1. to obtain the solution $u(x,t)$ in terms of $u_0(x)$ and $Q(x,t)$. You already know how the final formula must be.

Problem 2 [30 points] **Laplace Equation and Harmonic Functions**

You only must do one option for Part a. and one option for Part b. Clearly indicate the chosen ones. Only the two chosen will be graded.

Part a. Option I [20 points] Let Ω be a unit semidisk in $\mathbb{R}^2$, described in polar coordinates by $0 \leq r \leq R$, $0 \leq \theta \leq \pi$.

a.1) [10 points] Use the method of separation of variables to solve the Laplace equation in Ω with Neumann boundary conditions:

$$\Delta u(r,\theta) = \frac{\partial^2 u}{\partial r^2}(r,\theta) + \frac{1}{r}\frac{\partial u}{\partial r}(r,\theta) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}(r,\theta) = 0, \;\; 0 < r < R, 0 < \theta < \pi,$$
$$\frac{\partial u}{\partial \theta}(r,0) = \frac{\partial u}{\partial \theta}(r,\pi) = 0,$$
$$\frac{\partial u}{\partial r}(R,\theta) = f(\theta), \;\; |u(0,\theta)| < \infty.$$

a.2) [5 points] Use integration by parts to show that f must satisfy the condition

$$\int_0^\pi f(\theta) R d\theta = 0,$$

for a solution to exist. Physically, this condition means that the net heat flux across the boundary must be zero for an equilibrium temperature distribution to exist.

a.3) [5 points] Obtain a closed form formula for the solution, that is, one without a series in it (such as the Poisson formula).

Part a. Option II [20 points] This problem shows another application of Poisson's formula. We will find a proof for the so-called *Liouville's theorem* for harmonic functions: If $u(x,y)$ is a harmonic function in $\mathbb{R}^2$,

$$\Delta u(x,y) = 0, \;\; (x,y) \in \mathbb{R}^2, \tag{1}$$

and if u is bounded, $|u(x,y)| \leq M$, $M \in \mathbb{R}$, $(x,y) \in \mathbb{R}^2$, then u is constant.

a.1) [4 points] Show that the Laplace equation is invariant under translations. That is, show that if $u(x,y)$ solves (1), then $v(x,y) = u(x - x_c, y - y_c)$ also solves it, for any point $(x_c, y_c) \in \mathbb{R}^2$.

a.2) [4 points] For convenience, we will use the notation $\vec{x} = (x,y)$, $\vec{x}_c = (x_c, y_c)$. In the lecture notes (or book), we found the Poisson formula for harmonic functions in a disk centered at the origin. Of course, the translation invariance tell us that we could have done the same for a disk centered at any other point. In fact, let $v(\vec{x})$ be a harmonic function in a disk D_c centered at a point $\vec{x}_c \in \mathbb{R}^2$ with *arbitrary* radius R. It holds that for any $\vec{x} \in D_c$,

$$v(\vec{x}) = \frac{R^2 - |\vec{x} - \vec{x}_c|^2}{2\pi} \int_{|\vec{x}' - \vec{x}_c| = R} \frac{v(\vec{x}')}{|\vec{x} - \vec{x}'|^2} \frac{ds}{R}.$$

That is, Poisson' formula gives the value of a harmonic function at any point $\vec{x}$ inside of a disk in terms of its values on the boundary of that disk.

Show that

$$\nabla v(\vec{x}) = \frac{-2(\vec{x} - \vec{x}_c)}{2\pi} \int_{|\vec{x}' - \vec{x}_c| = R} \frac{v(\vec{x}')}{|\vec{x} - \vec{x}'|^2} \frac{ds}{R} + \frac{R^2 - |\vec{x} - \vec{x}_c|^2}{2\pi} \int_{|\vec{x}' - \vec{x}_c| = R} \frac{-2(\vec{x} - \vec{x}')v(\vec{x}')}{|\vec{x} - \vec{x}'|^4} \frac{ds}{R},$$

and thus

$$\nabla v(\vec{x}_c) = -\frac{R^2}{\pi} \int_{|\vec{x}' - \vec{x}_c| = R} \frac{(\vec{x}_c - \vec{x}')v(\vec{x}')}{|\vec{x}_c - \vec{x}'|^4} \frac{ds}{R},$$

a.3) [4 points] If $|v(\vec{x})| \le M$ for all $\vec{x} \in \mathbb{R}^2$, show that

$$|\nabla v(\vec{x}_c)| \le \frac{2}{R} M.$$

a.4) [4 points] Conclude the proof of Liouville's theorem.

a.5) [4 points] Use Liouville's theorem to prove the uniqueness of the following problem

$$\begin{aligned} \Delta u(x, y) &= f(x, y), \quad (x, y) \in \mathbb{R}^2, \\ |u(x, y)| &< \infty, \\ \lim_{|(x,y)| \to \infty} v(x, y) &= 0. \end{aligned}$$

Part b. Option I [10 points] This problem shows that many of the techniques we have studied for linear PDEs are also useful when dealing with nonlinear ones. Let Ω be an open, connected, bounded domain in $\mathbb{R}^2$, and $\partial\Omega$ be its boundary. Assume that u has continuous second derivatives in Ω and it is continuous in the closed set $\Omega \cup \partial\Omega$. Show that if u solves the *nonlinear* partial differential equation

$$\Delta u - u^2 = 0 \quad \text{in } \Omega,$$

then it cannot attain its maximum at an interior point unless $u \equiv 0$.

Part b. Option II [10 points] Define the function $u(x, y)$ as follows

$$u(x,y) = \begin{cases} \dfrac{1 - x^2 - y^2}{(1-x)^2 + y^2}, & (x,y) \neq (1,0), \\ 0, & (x,y) = (1,0). \end{cases}$$

b.1) [4 points] Prove that u is harmonic and positive in the disk $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$.

b.2) [2 points] What is the value of u at any point on the boundary circle $\partial D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$?

b.3) [4 points] Is this example a counterexample of the maximum principle?

Problem 3 [30 points] **Fourier Transform and Linear PDEs**

We have studied the heat, wave, and Laplace equation (prototypes of so-called *parabolic*, *hyperbolic*, and *elliptic* partial differential equations). In this problem we consider the linear Schrödinger equation for a free particle:

$$\begin{aligned} i\psi_t(x,t) + \psi_{xx}(x,t) &= 0, \quad x \in \mathbb{R}, t > 0, \\ \psi(x,0) &= \psi_0(x), \end{aligned}$$

with absolutely integrable initial data, i.e.,

$$\int_{-\infty}^{\infty} |\psi_0(x)| dx < \infty.$$

The function ψ is called the *wave or state function* in quantum mechanics, and defines the state of the physical system at each time instant.

This equation looks very similar to the heat equation, however the phenomena it describes are more similar to waves. It is an example of *dispersive* partial differential equations, which cannot be properly classified as either parabolic, hyperbolic, or elliptic. We will see that Fourier Transform methods are equally useful.

Part a. [6 points] Show that the Fourier transform of the solution is given by

$$\widehat{\psi}(\omega) = K(\omega,t)\widehat{\psi_0}(\omega),$$

where

$$K(\omega,t) = e^{-i\omega^2 t},$$

and thus the solution is

$$\psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega^2 t} e^{ix\omega}\widehat{\psi_0}(\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(-\omega t + x)}\widehat{\psi_0}(\omega)d\omega \tag{2}$$

Part b. [6 points] Show that for any real number $a > 0$,

$$\mathcal{F}\left(e^{iax^2}\right)(\omega) = \sqrt{\frac{\pi i}{a}} e^{-i\frac{\omega^2}{4a}},$$

where $\sqrt{i}$ denotes the root with positive real part, i.e., $\sqrt{i} = \frac{1+i}{\sqrt{2}}$. You can use without proof that

$$\int_{-\infty}^{\infty} \cos(x^2)dx = \sqrt{\frac{\pi}{2}}, \quad \int_{-\infty}^{\infty} \sin(x^2)dx = \sqrt{\frac{\pi}{2}}.$$

These are particular values of the so-called Fresnel integrals.

Part c. [6 points] Show that the solution is given by

$$\psi(x,t) = \frac{1}{\sqrt{4\pi i t}} \int_{-\infty}^{\infty} e^{i\frac{(x-y)^2}{4t}} \psi_0(y)dy.$$

Notice the similarity of this formula with the one for the heat equation.

Part d. [6 points] Prove the *dispersive* estimate

$$\max_{x\in\mathbb{R}} |\psi(x,t)| \leq \frac{C}{t^{1/2}},$$

where $C \in \mathbb{R}$ is a constant that depends on the initial data. That is, the amplitude of the wave decreases in time.

Part e. [6 points] Use Parseval's theorem (also called Plancherel's) to prove the preservation in time of the L^2 norm

$$\|\psi(t,\cdot)\|_{L^2(\mathbb{R})} = \|\psi_0\|_{L^2(\mathbb{R})},$$

that is, the total energy is preserved in time.

Remark: This does not contradict Part d. The maximum of ψ decreases but its support will increase.

Part f. [Optional, no points] The phenomenon that explains Parts d.-e. is called *dispersion*: waves with different frequencies travel at different velocities. That is how the support of the solution gets spread out in space. Look at the last term in formula (2): the solution ψ is written as a (continuum) superposition of waves $e^{i\omega(-\omega t+x)}$, each traveling at its own velocity ω. Check that dispersion does not happen in the 1D wave equation, i.e., the velocity of the waves are independent of their frequency.