

Clases 23, 24 = Ejercicios bloque I.

- Ejemplo 4: Extremos absolutos de $f(x, y) = xy^2$ en la región

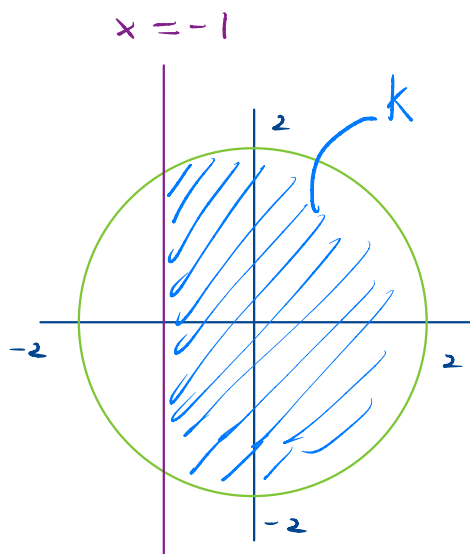
$$K = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4, x \geq -1\}$$

Sol:

Representamos el conjunto K :

$$x^2 + y^2 \leq 4 \rightarrow \text{círculo radio 2}$$

$$x \geq -1 \rightarrow \text{semiplano}$$



Como K es acotado y cerrado y f continua en K , por T^{te} Weierstrass f alcanza máx. / mín. abs.
¿Candidatos?

- 1) Extremos relativos en el interior de K .
- 2) Extremos condicionados en la frontera dada por $x = -1$.

3) Extremos condicionados a la frontera circular,
 $x^2 + y^2 = 4$.

4) Extremos en los vértices de K , i.e., puntos de intersección de la frontera.

$$1) \nabla f(x, y) = (y^2, 2xy) = 0 \Leftrightarrow \begin{matrix} y^2 = 0 \\ 2xy = 0 \end{matrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} y = 0 \\ x \in (-1, 2) \end{cases} \rightarrow (x_0, 0) \text{ con } x_0 \in (-1, 2)$$

2) Por sustitución: $f(-1, y) = -y^2 \rightarrow$

$$\rightarrow \frac{d}{dy} (f(-1, y)) = -2y = 0 \Leftrightarrow y = 0 \rightarrow (-1, 0)$$

$$3) L(x, y, \lambda) = xy^2 + \lambda(x^2 + y^2 - 4)$$

$$\nabla L(x, y, \lambda) = \begin{pmatrix} y^2 + 2\lambda x \\ 2xy + 2\lambda y \\ x^2 + y^2 - 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} y^2 + 2\lambda x = 0 \\ 2y(x + \lambda) = 0 \\ x^2 + y^2 = 4 \end{cases} \begin{cases} \textcircled{1} y = 0 \\ \textcircled{2} x = -\lambda \end{cases}$$

$$\textcircled{1} \quad \begin{matrix} y=0 \\ 2\lambda x=0 \\ x^2=4 \end{matrix} \left\{ \begin{array}{l} \xrightarrow{\quad} \lambda=0 \rightarrow \begin{matrix} (2,0) \\ (-2,0) \notin K \end{matrix} \\ \wedge x=\pm 2 \end{array} \right.$$

$$\textcircled{2} \quad \begin{matrix} x=-\lambda \\ y^2-2\lambda^2=0 \\ \lambda^2+y^2=4 \end{matrix} \left\{ \begin{array}{l} \xleftarrow{\quad} y^2=2\lambda^2 \\ \wedge \lambda^2+y^2=4 \end{array} \right. \left\{ \begin{array}{l} \xrightarrow{\quad} y=\pm \frac{2\sqrt{2}}{\sqrt{3}} \\ \wedge 3\lambda^2=4 \rightarrow \lambda=\pm \frac{2}{\sqrt{3}} \end{array} \right.$$

$$\left(\frac{2}{\sqrt{3}}, \pm \frac{2\sqrt{2}}{\sqrt{3}} \right), \left(-\frac{2}{\sqrt{3}}, \pm \frac{2\sqrt{2}}{\sqrt{3}} \right) \notin K$$

$$4) \text{ Vertices: } \begin{matrix} x=-1 \\ x^2+y^2=4 \end{matrix} \rightarrow y^2=3 \rightarrow y=\pm\sqrt{3} \rightarrow (-1, \pm\sqrt{3})$$

• Evaluamos: $f(x_0, 0) = 0 \quad \forall x_0 \in (-1, 2)$

$$f(-1, 0) = 0$$

$$f(2, 0) = 0$$

$$f\left(\frac{2}{\sqrt{3}}, \pm \frac{2\sqrt{2}}{\sqrt{3}}\right) = \frac{2}{\sqrt{3}} \cdot \frac{8}{3} \rightarrow \text{Máx. abs.}$$

$$f(-1, \pm\sqrt{3}) = -3 \rightarrow \text{Mín. abs.}$$

• Ejercicios: Hoja 3: 25, 20, 19

Hoja 2: 21,

35, 40, 25

27, 28, 29, 30

25 (Hoj-3)

$$\begin{array}{l} \min/\max. d(x,y,z) \\ \text{s.t.} \quad \left\{ \begin{array}{l} x^2 + y^2 + z^2 = 1 \\ x + y + z = 1 \end{array} \right. \end{array} \quad \left\{ \begin{array}{l} \text{con } d(x,y,z) = \|(x,y,z) - (0,3,3)\| \\ \downarrow \\ d(x,y,z)^2 = x^2 + (y-3)^2 + (z-3)^2 \end{array} \right.$$

(*)

$$\begin{aligned} L(x,y,z,\lambda,\mu) = & x^2 + (y-3)^2 + (z-3)^2 + \\ & + \lambda(x^2 + y^2 + z^2 - 1) + \mu(x + y + z - 1). \end{aligned}$$

Resolvemos $\nabla L = \vec{0}$:

$$\left\{ \begin{array}{l} 2x + 2\lambda x + \mu = 0 \\ 2(y-3) + 2\lambda y + \mu = 0 \\ 2(z-3) + 2\lambda z + \mu = 0 \\ x^2 + y^2 + z^2 = 1 \\ x + y + z = 1 \end{array} \right. \rightarrow \begin{array}{l} \rightarrow z = 1 - x - y \end{array}$$

$$\left\{ \begin{array}{l} 2x + 2\lambda x + \mu = 0 \\ 2(y-3) + 2\lambda y + \mu = 0 \\ 2(-2-x-y) + 2\lambda(1-x-y) + \mu = 0 \\ x^2 + y^2 + (1-x-y)^2 = 1 \end{array} \right. \rightarrow \begin{array}{l} \mu = -2x - 2\lambda x \\ 2(y-3) + 2\lambda y = 2x + 2\lambda x \\ -4 - 2x - 2y + 2\lambda(1-x-y) = 2x + 2\lambda x \\ x^2 + y^2 + (1-x-y)^2 = 1 \end{array}$$

$$\left. \begin{array}{l} 1) \quad 2y - 2x + 2\lambda y - 2\lambda x - 6 = 0 \\ 2) \quad -2y - 4x - 2\lambda y - 4\lambda x - 4 + 2\lambda = 0 \\ 3) \quad x^2 + y^2 + (1-x-y)^2 = 1 \end{array} \right\} \rightarrow$$

$$\text{II}_{1+2):} \quad -6x - 6\lambda x - 10 + 2\lambda = 0 \rightarrow$$

$$\rightarrow 2\lambda(1-3x) = 10 + 6x \rightarrow \lambda = \frac{5+3x}{1-3x}$$

(x ≠ 1/3) ⊥

$$\left. \begin{array}{l} 2y - 2x + 2y \frac{3x+5}{1-3x} - 2x \frac{3x+5}{1-3x} - 6 = 0 \\ x^2 + y^2 + 1 + x^2 + y^2 - 2x - 2y + 2xy = 1 \end{array} \right\} \rightarrow$$

$$\rightarrow 2y - 6xy - 2x + 6x^2 + 6xy + 10y - 6x^2 - 10x - 6 + 18x = 0$$

$$2x^2 + 2y^2 - 2x - 2y + 2xy = 0$$

$$2y + x - 1 = 0 \quad \rightarrow x = 1 - 2y$$

$$\rightarrow x^2 + y^2 - x - y + xy = 0 \quad \left(1 + 4y^2 - 4y + y^2 - 1 + 2y - y + y - 2y^2 = 0 \right)$$

$$\rightarrow 3y^2 - 2y = 0 \rightarrow y(3y - 2) = 0$$

$\nearrow y = 0 \rightarrow x = 1$
 $\searrow y = \frac{2}{3} \rightarrow x = -\frac{1}{3}$

Así,

1) $(1, 0, 0)$

2) $(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$

Dado que la circunferencia de intersección es un conjunto compacto, por T^a Weierstrass existen los máx./mín. absolutos.

Evaluamos:

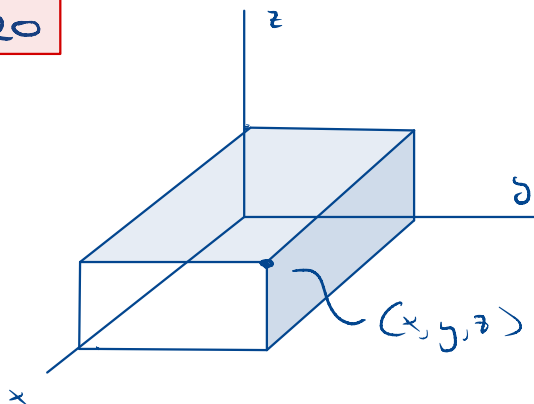
$$1^2 + (0-3)^2 + (0-3)^2 = 19$$

$$\frac{1}{3^2} + \left(\frac{2}{3}-3\right) \cdot 2 = \frac{1}{9} + \frac{7^2}{3^2} \cdot 2 = \frac{1+98}{9} = 11$$

Máx. abs. en $(1, 0, 0)$

→ Míx. abs. en $(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$

20



$$V = xyz$$

$$\text{máx. } xyz$$

$$\text{s.t. } 6x + 4y + 3z = 24 \quad \left\{ \right.$$

$$\mathcal{L}(x, y, z) = xyz + \lambda(6x + 4y + 3z - 24)$$

$$\nabla \mathcal{L} = \vec{0} \rightarrow$$

$$\begin{aligned}
 yz + 6x &= 0 \\
 xz + 4y &= 0 \\
 xy + 3z &= 0 \\
 6x + 4y + 3z - 24 &= 0
 \end{aligned}
 \quad \rightarrow \quad
 \begin{cases}
 -\frac{yz}{6} = -\frac{xz}{4} \\
 xz + 4y = 0 \\
 \rightarrow \lambda = -\frac{1}{3}xy \\
 \rightarrow z = 8 - \frac{4}{3}y - 2x
 \end{cases}$$

$$\begin{aligned}
 \textcircled{1} z &= 0 \\
 \textcircled{2} -2y + 3x &= 0
 \end{aligned}$$

$$\begin{aligned}
 \textcircled{1} z &= 0 \\
 \lambda &= 0 \\
 xy &= 0 \\
 24 - 4y - 6x &= 0
 \end{aligned}
 \quad \left\{ \begin{aligned}
 x &= 0 \rightarrow y = 6 \rightarrow (0, 6, 0) \\
 y &= 0 \rightarrow x = 4 \rightarrow (4, 0, 0)
 \end{aligned} \right.$$

$$\begin{aligned}
 \textcircled{2} 3x &= 2y \\
 xz - \frac{4}{3}xy &= 0 \\
 \lambda &= -xy/3 \\
 z &= 8 - \frac{4}{3}y - 2x
 \end{aligned}
 \quad \left\{ \begin{aligned}
 \textcircled{2.1} x &= 0, y = 0, z = 8 \rightarrow (0, 0, 8) \\
 \textcircled{2.2} 3z &= 4y
 \end{aligned} \right.$$

$$\begin{aligned}
 \textcircled{2.2} 3x &= 2y \\
 3z &= 4y \\
 3z &= 24 - 4y - 6x \\
 \lambda &= -xy/3
 \end{aligned}
 \quad \rightarrow \quad
 \begin{cases}
 3x = 2y \\
 4y = 24 - 4y - 6x \\
 z = \frac{4}{3}y \\
 \lambda = -xy/3
 \end{cases}
 \quad \left\{ \begin{aligned}
 3x - 2y &= 0 \\
 3x + 2y &= 24
 \end{aligned} \right. \rightarrow$$

$$\rightarrow x = 4, y = 6, z = 8 \rightarrow (4, 6, 8)$$

$$\text{Volume max} = 4 \cdot 6 \cdot 8$$

19. máx. $x^2 y^2 z^2$ $\left\{ \begin{array}{l} \text{Probar que } \forall a, b, c > 0, \\ \text{s.a. } x^2 + y^2 + z^2 = r^2 \end{array} \right. \quad (abc)^{1/3} \leq \frac{1}{3}(a+b+c),$

Sol:

$$L(x, y, z, \lambda) = x^2 y^2 z^2 + \lambda (x^2 + y^2 + z^2 - r^2)$$

$$\left\{ \begin{array}{l} 2x y^2 z^2 + 2\lambda x = 0 \\ 2y x^2 z^2 + 2\lambda y = 0 \\ 2z x^2 y^2 + 2\lambda z = 0 \\ x^2 + y^2 + z^2 = r^2 \end{array} \right. \quad \left\{ \begin{array}{l} \textcircled{1} x=0 \\ \textcircled{2} y^2 z^2 + \lambda = 0 \end{array} \right.$$

$$\textcircled{1} \quad \left\{ \begin{array}{l} x=0 \\ \lambda y=0 \\ \lambda z=0 \\ y^2 + z^2 = r^2 \end{array} \right. \quad \left\{ \begin{array}{l} \textcircled{1.1} \lambda=0 \\ \textcircled{1.2} y=0 \end{array} \right.$$

$$\textcircled{1.1} \quad x=0, \lambda=0 \rightarrow \{(x, y, z) \in \mathbb{R}^3 : x=0, y^2 + z^2 = r^2\}$$

$$\textcircled{1.2} \quad \left\{ \begin{array}{l} x=0, y=0 \\ \lambda z=0 \\ z^2 = r^2 \end{array} \right. \quad \left\{ \begin{array}{l} \rightarrow x=0, y=0, z=\pm r \\ \lambda=0 \end{array} \right.$$

$$\textcircled{2} \quad \left\{ \begin{array}{l} y^2 z^2 + \lambda = 0 \\ y(x^2 z^2 + \lambda) = 0 \\ z(x^2 y^2 + \lambda) = 0 \\ x^2 + y^2 + z^2 = r^2 \end{array} \right. \quad \left\{ \begin{array}{l} \textcircled{2.1} y=0 \\ \textcircled{2.2} x^2 z^2 + \lambda = 0 \end{array} \right.$$

$$\textcircled{2.1} \quad \lambda = 0, y = 0 \quad \left\{ \rightarrow \{(x, y, z) \in \mathbb{P}^3 : y = 0, x^2 + z^2 = r^2\} \right.$$

$$x^2 + z^2 = r^2$$

$$\textcircled{2.2} \quad \begin{cases} y^2 z^2 = -\lambda \\ x^2 z^2 = -\lambda \\ z(x^2 y^2 + \lambda) = 0 \\ x^2 + y^2 + z^2 = r^2 \end{cases} \quad \left\{ \begin{array}{l} \textcircled{2.2.1} \quad z = 0 \\ \textcircled{2.2.2} \quad x^2 y^2 + \lambda = 0 \end{array} \right.$$

$$\textcircled{2.2.1} \quad \lambda = 0 \quad \left\{ \rightarrow \{(x, y, z) \in \mathbb{P}^3 : z = 0, x^2 + y^2 = r^2\} \right.$$

$$x^2 + y^2 = r^2$$

$$\textcircled{2.2.2} \quad \begin{cases} y^2 z^2 = -\lambda \\ x^2 z^2 = -\lambda \\ x^2 y^2 = -\lambda \\ x^2 + y^2 + z^2 = r^2 \end{cases} \quad \left\{ \begin{array}{l} y^2 z^2 = -\lambda \\ x^2 z^2 = y^2 z^2 \\ x^2 y^2 = y^2 z^2 \\ x^2 + y^2 + z^2 = r^2 \end{array} \right. \quad \left\{ \begin{array}{l} \textcircled{2.2.2.1} \quad z = 0 \quad \left\{ \begin{array}{l} x = 0, y = \pm r \\ y = 0, x = \pm r \end{array} \right. \\ \textcircled{2.2.2.2} \quad x = \pm z \end{array} \right.$$

$$\textcircled{2.2.2.2} \quad \begin{cases} \lambda = -y^2 z^2 \\ x = \pm z \\ y^4 = y^2 z^2 \\ 2y^2 + z^2 = r^2 \end{cases} \quad \left\{ \begin{array}{l} y = 0, x = 0, z = \pm r \\ y = \pm z \rightarrow 3y^2 = r^2 \rightarrow y = \pm \sqrt{\frac{r^2}{3}} \\ x = \pm \sqrt{\frac{r^2}{3}} \\ z = \pm \sqrt{\frac{r^2}{3}} \end{array} \right.$$

$$\text{Máx. abs.} = r \left(\pm \frac{1}{3}, \pm \frac{1}{3}, \pm \frac{1}{3} \right), \text{ con valor } \left(\frac{r^2}{3} \right)^3$$

$$\bullet \quad d(abc)^{\frac{1}{3}} \leq \frac{1}{3}(a+b+c) \quad \forall a, b, c > 0?$$

Dados $a, b, c > 0$, denotemos $x^2 = a, y^2 = b, z^2 = c$.
Denotamos $r^2 = a + b + c$.

Entonces, por el apartado anterior,

$$a \cdot b \cdot c \leq \left(\frac{a+b+c}{3} \right)^3 \Rightarrow (a \cdot b \cdot c)^{1/3} \leq \frac{1}{3} (a+b+c) //$$

21 (Hoja 2)

$$z = \sqrt{4 - x^2 - y^2}. \quad (x, y) \text{ de } (1, 1) \text{ a } (1.01, 0.99).$$

Aprox. lineal:

$$z(x, y) \simeq z_0 + \nabla z(x_0, y_0) \cdot (x - x_0, y - y_0),$$

$$\nabla z(x, y) = \left(\frac{-2x}{2\sqrt{4-x^2-y^2}}, \frac{-2y}{2\sqrt{4-x^2-y^2}} \right).$$

$$\Rightarrow z(1.01, 0.99) \simeq \sqrt{2} + \left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right) \cdot (0.01, -0.03) \simeq 1.433$$

$$\text{Valor exacto: } z(1.01, 0.99) = \sqrt{4 - 1.01^2 - 0.99^2} \simeq 1.428$$

35 $\Rightarrow$ Escribir la ecuación $\Delta u = \partial_x^2 u + \partial_y^2 u = 0$
en polares.

Sl: ∇^2 Ec. para $\tilde{u}$ con $\tilde{u}(r, \theta) = u(x(r, \theta), y(r, \theta)) //$

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \end{aligned} \quad \rightarrow \quad \begin{aligned} \partial_r x &= \cos \theta, \quad \partial_\theta x = -r \sin \theta \\ \partial_r y &= \sin \theta, \quad \partial_\theta y = r \cos \theta \end{aligned}$$

Chain rule:

$$\bullet \quad \frac{\partial \tilde{u}}{\partial r} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r} = \partial_x u \cos \theta + \partial_y u \sin \theta$$

$$\bullet \quad \frac{\partial^2 \tilde{u}}{\partial r^2} = \cos \theta \frac{\partial}{\partial r} \partial_x u + \sin \theta \frac{\partial}{\partial r} \partial_y u =$$

$$= \cos \theta \left(\frac{\partial}{\partial x} (\partial_x u) \frac{\partial x}{\partial r} + \frac{\partial}{\partial y} (\partial_x u) \frac{\partial y}{\partial r} \right) +$$

$$+ \sin \theta \left(\frac{\partial}{\partial x} (\partial_y u) \frac{\partial x}{\partial r} + \frac{\partial}{\partial y} (\partial_y u) \frac{\partial y}{\partial r} \right) =$$

$$= \cos^2 \theta \partial_{xx}^2 u + 2 \cos \theta \sin \theta \partial_{xy}^2 u + \sin^2 \theta \partial_{yy}^2 u.$$

De igual forma obtenemos que

$$\bullet \quad \frac{\partial \tilde{u}}{\partial \theta} = \dots = -r \sin \theta \partial_x u + r \cos \theta \partial_y u,$$

$$\bullet \quad \frac{\partial^2 \tilde{u}}{\partial \theta^2} = \dots = -r \left(\overbrace{\cos \theta \partial_x u + \sin \theta \partial_y u}^{\partial_r u} \right) +$$

$$+ r^2 (\sin^2 \theta \partial_{xx}^2 u - 2 \cos \theta \sin \theta \partial_{xy}^2 u + \cos^2 \theta \partial_{yy}^2 u).$$

$$\hookrightarrow \partial_{\theta\theta}^2 \tilde{u} = -r \partial_r u + r^2 \sin^2 \theta \partial_{xx}^2 u + r^2 \cos^2 \theta \partial_{yy}^2 u - 2r^2 \cos \theta \sin \theta \partial_{xy}^2 u.$$

Finalmente, sumamos las expresiones de $\nabla_r^2 \tilde{u}$ y $\frac{1}{r^2} \nabla_{\theta}^2 \tilde{u}$:

$$\frac{\partial^2 \tilde{u}}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \tilde{u}}{\partial \theta^2} = -\frac{1}{r} \frac{\partial \tilde{u}}{\partial r} + \frac{\partial^2 \tilde{u}}{\partial x^2} + \frac{\partial^2 \tilde{u}}{\partial y^2} \Rightarrow$$

$$\Rightarrow \boxed{\frac{\partial^2 \tilde{u}}{\partial x^2} + \frac{\partial^2 \tilde{u}}{\partial y^2} = \frac{\partial^2 \tilde{u}}{\partial r^2} + \frac{1}{r} \frac{\partial \tilde{u}}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \tilde{u}}{\partial \theta^2}}$$

Laplacian
in polar
coordinates.

40 $w(x,t) = \tilde{w}(u(x,t), v(x,t)).$

$$\frac{\partial w}{\partial x} = \frac{\partial \tilde{w}}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial \tilde{w}}{\partial v} \frac{\partial v}{\partial x} = \frac{\partial \tilde{w}}{\partial u} + \frac{\partial \tilde{w}}{\partial v}$$

$$\frac{\partial w}{\partial t} = \frac{\partial \tilde{w}}{\partial u} \frac{\partial u}{\partial t} + \frac{\partial \tilde{w}}{\partial v} \frac{\partial v}{\partial t} = -a \frac{\partial \tilde{w}}{\partial u} + a \frac{\partial \tilde{w}}{\partial v}$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial \tilde{w}}{\partial u} \right) + \frac{\partial}{\partial x} \left(\frac{\partial \tilde{w}}{\partial v} \right) =$$

$$= \frac{\partial^2 \tilde{w}}{\partial u^2} \frac{\partial u}{\partial x} + \frac{\partial^2 \tilde{w}}{\partial u \partial v} \frac{\partial v}{\partial x} + \frac{\partial^2 \tilde{w}}{\partial v \partial u} \frac{\partial u}{\partial x} + \frac{\partial^2 \tilde{w}}{\partial v^2} \frac{\partial v}{\partial x} =$$

$$= \partial_{uu}^2 \tilde{w} + 2 \partial_{uv}^2 \tilde{w} + \partial_{vv}^2 \tilde{w},$$

$$\frac{\partial^2 w}{\partial t^2} = -a \left(\frac{\partial^2 \tilde{w}}{\partial u^2} \frac{\partial u}{\partial t} + \frac{\partial^2 \tilde{w}}{\partial u \partial v} \frac{\partial v}{\partial t} \right) + a \left(\frac{\partial^2 \tilde{w}}{\partial v \partial u} \frac{\partial u}{\partial t} + \frac{\partial^2 \tilde{w}}{\partial v^2} \frac{\partial v}{\partial t} \right) =$$

$$= -a \left(-a \partial_{uu}^2 \tilde{w} + a \partial_{uv}^2 \tilde{w} \right) + a \left(-a \partial_{uv}^2 \tilde{w} + a \partial_{vv}^2 \tilde{w} \right) =$$

$$= a^2 \partial_{uu}^2 \tilde{w} - 2a^2 \partial_{uv}^2 \tilde{w} + a^2 \partial_{vv}^2 \tilde{w}$$

Entonces, $\nabla_{tt}^2 w = a^2 \nabla_{xx}^2 w \Rightarrow$

$$\Rightarrow a^2 \nabla_{uu}^2 \tilde{w} - 2a^2 \nabla_{uv}^2 \tilde{w} + a^2 \nabla_{vv}^2 \tilde{w} =$$

$$= a^2 \nabla_{uu}^2 \tilde{w} + 2a^2 \nabla_{uv}^2 \tilde{w} + a^2 \nabla_{vv}^2 \tilde{w} \Rightarrow$$

$$\Rightarrow \boxed{\nabla_{uv}^2 \tilde{w} = 0}$$

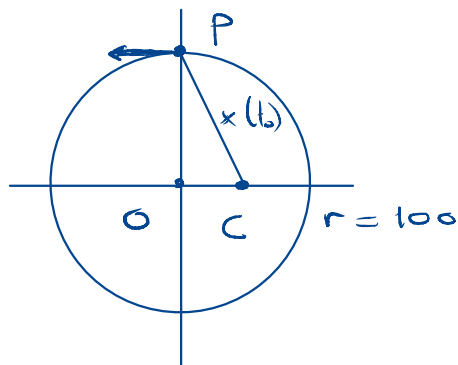
Note: Solución: $\nabla_{uv}^2 \tilde{w} = 0 \Rightarrow \nabla_v \tilde{w} = f_1(u) \Rightarrow$

$$\Rightarrow \tilde{w} = g_1(u) + g_2(v) \Rightarrow$$

$$\Rightarrow \boxed{w(x,t) = g_1(x+at) + g_2(x-at)}$$

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$$V(t) = \frac{c(t)}{x(t)}$$

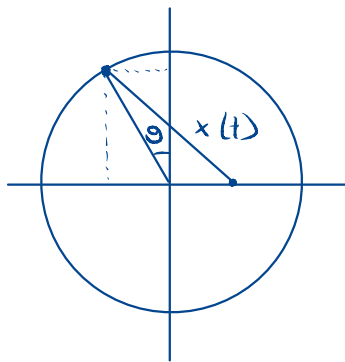


En el instante pedido:

$$\left. \begin{array}{l} c(t_0) = 5 \\ \dot{c}(t_0) = 1 \end{array} \right\} \dot{V}(t_0) ?$$

$$\dot{V}(t) = \frac{\dot{c}(t)x(t) - c(t)\dot{x}(t)}{(x(t))^2} \leftarrow d \ x(t), \dot{x}(t) ?$$

Obtenemos la expresión de la distancia $x(t)$:



$$(x(t))^2 = \left(\frac{r}{2} + r \cdot \sin \theta(t)\right)^2 + (r \cdot \cos \theta(t))^2$$

$$\dot{\theta}(t) = 1, \theta(t_0) = 0.$$

Entonces:

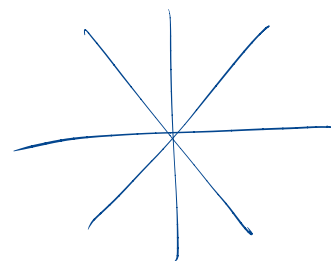
$$2x \cdot \dot{x} = 2\left(\frac{r}{2} + r \cdot \sin \theta\right) r \cdot \cos \theta \dot{\theta} + 2r \cdot \cos \theta (-\sin \theta) \dot{\theta}$$

Como $x(t_0) = \sqrt{r^2 + \left(\frac{r}{2}\right)^2}$, tenemos todas las
datos para despejar $\dot{x}(t_0)$ y sustituir en $\dot{V}(t_0)$.

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$$f(x, y) = \begin{cases} \frac{x^3 - y^3}{x^2 - y^2} & |x| \neq |y|, \\ \frac{3}{2}x & |x| = |y|. \end{cases}$$

- Continuidad en $\mathbb{R}^2$.
- Calcular $J_x f, J_y f$.
- Derivadas direccionales en $(0,0)$.
- Diferenciabilidad.



Sol:

a)

Estudiamos la continuidad en las rectas $|x|=|y|$.

Primero notamos que:

$$x^2 - y^2 = (x+y)(x-y),$$

$$x^3 - y^3 = (x^2 + xy + y^2)(x-y)$$

1) Punto (x_0, x_0) , $x_0 \neq 0$:

$$\lim_{(x,y) \rightarrow (x_0, x_0)} \frac{x^3 - y^3}{x^2 - y^2} = \lim_{(x,y) \rightarrow (x_0, x_0)} \frac{x^2 + xy + y^2}{x+y} = \frac{3x_0^2}{2x_0} = \frac{3}{2}x_0$$

$\Rightarrow$ Es continua en (x_0, x_0) , $x_0 \neq 0$.

2) Punto $(x_0, -x_0)$, $x_0 \neq 0$:

$$\lim_{(x,y) \rightarrow (x_0, -x_0)} \frac{x^2 + xy + y^2}{x+y} = \left(\frac{x_0^2}{0} \right) = \infty \Rightarrow \text{No es cont.}$$

3) En $(0,0)$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + xy + y^2}{x+y}$$

Vemos que en $y \sim -x$ el denominador se hace cero.

Estudiamos el límite a lo largo de

$$y = -x + mx^2:$$

$$\lim_{\substack{x \rightarrow 0 \\ y = -x + mx^2}} \frac{x^2 + xy + y^2}{x+y} =$$

$$= \lim_{x \rightarrow 0} \frac{\cancel{x^2} - \cancel{x^2} + mx^3 + (-x + mx^2)^2}{mx^2} = \lim_{x \rightarrow 0} \frac{mx + (-1 + mx)^2}{m} =$$

$$= \frac{-1}{m} \Rightarrow \text{El lím. doble no existe} \Rightarrow \text{No es cont. en } (0,0).$$

• f es continua en $\mathbb{R}^2 \setminus \{(x_0, -x_0) : x_0 \in \mathbb{R}\}$

b) Si $|x| \neq |y|$:

$$\begin{aligned} \partial_x f(x,y) &= \frac{1}{\partial x} \left(\frac{x^2 + xy + y^2}{x+y} \right) = \frac{(2x+y)(x+y) - (x^2 + xy + y^2)}{(x+y)^2} = \\ &= \frac{x^2 + 2xy}{(x+y)^2}, \end{aligned}$$

$$\partial_y f(x,y) = \frac{(x+2y)(x+y) - (x^2 + xy + y^2)}{(x+y)^2} = \frac{2xy + y^2}{(x+y)^2}.$$

• En $x = -y \neq 0$:

$$\partial_x f(x, -x) = \lim_{h \rightarrow 0} \frac{f(x+h, -x) - f(x, -x)}{h} =$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{(x+h)^2 - x(x+h) + x^2}{h} - \frac{3}{2}x \right) =$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{\cancel{2x^2} + 2h^2 + 4xh - \cancel{2x^2} - 2xh + 2x^2 - 3xh}{2h} \right) =$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left(h - \frac{x}{2} + \frac{x^2}{h} \right) = \infty$$

$$\Delta_y f(x, -x) = \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{x^2 + x(-x+h) + (x-h)^2}{h} - \frac{3}{2}x \right) =$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{\cancel{x^2} - \cancel{x^2} + xh + x^2 + h^2 - 2xh}{h} - \frac{3x}{2} \right) = \infty.$$

• En $x = y \neq 0$:

$$\Delta_x f(x, x) = \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{(x+h)^2 + (x+h)x + x^2}{2x+h} - \frac{3}{2}x \right) =$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \frac{\cancel{2x^2} + 2h^2 + 4xh + \cancel{2x^2} + 2xh + \cancel{2x^2} - \cancel{6x^2} - 3xh}{2(2x+h)} =$$

$$= \frac{3x}{4x} = \frac{3}{4}.$$

$$\Delta_y f(x, x) = \frac{3}{4}.$$

• En $x = y = 0$:

$$\Delta_x f(0, 0) = \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \frac{h^2}{h} = 1,$$

$$\Delta_y f(0, 0) = \lim_{h \rightarrow 0} \frac{1}{h} \frac{h^2}{h} = 1.$$

c) Si $\vec{u} = (u_1, u_2)$ con $u_1^2 + u_2^2 = 0$, $u_1 + u_2 \neq 0$,

$$\begin{aligned} D_{\vec{u}} f(0,0) &= \lim_{h \rightarrow 0} \frac{f(hu_1, hu_2) - f(0,0)}{h} = \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \frac{(hu_1)^2 + h^2 u_1 u_2 + h^2 u_2^2}{h(u_1 + u_2)} = \frac{u_1^2 + u_1 u_2 + u_2^2}{u_1 + u_2} \end{aligned}$$

d) f es diferenciable en $\mathbb{R}^2 \setminus \{(x,y) \in \mathbb{R}^2 : |x| = |y|\}$

En $(x, -x)$, $x \in \mathbb{R}$, no es diferenciable puesto que no es continua

En $(x, x) \neq (0,0)$ sí es diferenciable, pues existen las parciales en un entorno del punto y son continuas en el punto:

$$\partial_x f(x,y) = \frac{x^2 + 2xy}{(x+y)^2} \quad \text{si } |x| \neq |y|,$$

$$\partial_y f(x,y) = \frac{2xy + y^2}{(x+y)^2}$$

para $x_0 \neq 0$:

$$\lim_{(x,y) \rightarrow (x_0, x_0)} \partial_x f(x,y) = \frac{3}{4} = \partial_x f(x_0, x_0),$$

$$\lim_{(x,y) \rightarrow (x_0, x_0)} \frac{2xy + y^2}{(x+y)^2} = \frac{3}{4} = \partial_y f(x_0, x_0).$$

27

$$a) D(f) = \{(x, y) \in \mathbb{R}^2 : 2x^2 - y^2 - xy \neq 0\} \cup \{(0, 0)\}.$$

$$p(x, y) = 2x^2 - y^2 - xy$$

Vemos que $p(x, x) = 0$, es decir, $y = x$ es una raíz:

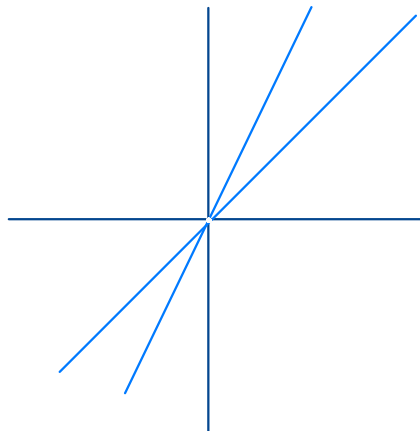
$$\begin{array}{r} 2x^2 - y^2 - xy \\ -2x^2 + 2xy \\ \hline -y^2 + xy \\ y^2 - xy \end{array} \quad \frac{y-x}{-2x-y} \Rightarrow p(x, y) = -(y-x)(2x+y)$$

$$\begin{aligned} \text{También: } 2x^2 - y^2 - xy &= x^2 - y^2 + x^2 - xy = \\ &= (x-y)(x+y) + (x-y)x = (x-y)(2x+y) \end{aligned}$$

$$p(x, y) = 0 \Leftrightarrow y = x \text{ o } y = -2x$$

$$D(f) = \mathbb{R}^2 \setminus \left(\{(x, y) \in \mathbb{R}^2 : y = x, x \neq 0\} \cup \{(x, y) \in \mathbb{R}^2 : y = -2x, x \neq 0\} \right).$$

Gráficamente:



El plano menos las
dos rectas, y el origen.

$$b) \lim_{\substack{x \rightarrow 0 \\ y = x + x^2}} \frac{x^3}{(x-y)(2x+y)} = \lim_{x \rightarrow 0} \frac{x^3}{-x^2(3x+x^2)} = -\frac{1}{3}$$

c) No es continua en $(0,0)$ pues

$$\lim_{\substack{y \rightarrow 0 \\ x=0}} \frac{x^3}{2x^2 - y^2 - xy} = 0 \neq -\frac{1}{3}.$$

Por tanto, tampoco es diferenciable en $(0,0)$.

d)

$$\frac{df}{dx}(0,0) = \lim_{h \rightarrow 0} \frac{f(h,0) - 0}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \frac{h^3}{2h^2} = \frac{1}{2}$$

$$\frac{d^2f}{dx dy}(0,0) = \lim_{h \rightarrow 0} \frac{d_y f(h,0) - d_y f(0,0)}{h},$$

$$d_y f(0,0) = \lim_{h \rightarrow 0} \frac{f(0,h) - f(0,0)}{h} = 0,$$

$$d_y f(x,y) = \frac{-x^3(-2y-x)}{(2x^2 - y^2 - xy)^2} \quad \text{si } (x,y) \neq (0,0)$$

$$\Rightarrow \frac{d^2f}{dx dy}(0,0) = \lim_{h \rightarrow 0} \frac{1}{h} \frac{-h^3(-h)}{(2h^2)^2} = \frac{1}{4}$$

Clases 25, 26: Control en clase

1.

a) Continua en $\mathbb{R}^2 \setminus \{(0,0)\}$.

En $(0,0)$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x\sqrt{|y|}}{\sqrt{x^2+y^2}} = 0 \text{ pues}$$

$$0 \leq \left| \frac{x\sqrt{|y|}}{\sqrt{x^2+y^2}} \right| \leq \sqrt{\frac{x^2}{x^2+y^2}} \sqrt{|y|} \leq \sqrt{|y|} \rightarrow 0.$$

Usando polares también:

$$\frac{x\sqrt{|y|}}{\sqrt{x^2+y^2}} = \frac{r^{3/2} \cos \theta \sqrt{|\sin \theta|}}{r} = r^{1/2} \cos \theta \sqrt{|\sin \theta|}$$

Como $|r^{1/2} \cos \theta \sqrt{|\sin \theta|}| \leq r^{1/2}$ $\forall \theta \in [0, 2\pi)$ y $\lim_{r \rightarrow 0^+} r^{1/2} = 0$,

concluimos que $\lim_{(x,y) \rightarrow (0,0)} \frac{x\sqrt{|y|}}{\sqrt{x^2+y^2}} = 0$.

b) Diferenciable y C^1 en $\mathbb{R}^2 \setminus \{(0,0)\}$. En $(0,0)$:

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \rightarrow 0} \frac{f(h,0) - f(0,0)}{h} = 0 = \lim_{h \rightarrow 0} \frac{f(0,h) - f(0,0)}{h} = \frac{\partial f}{\partial y}(0,0)$$

$$\Rightarrow \nabla f(0,0) = (0,0).$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{f(x,y) - f(0,0) - \nabla f(0,0) \cdot (x-0, y-0)}{\|(x,y) - (0,0)\|} =$$

$$= \lim_{(x,y) \rightarrow (0,0)} \frac{\frac{x\sqrt{|y|}}{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}} = \lim_{(x,y) \rightarrow (0,0)} \frac{x\sqrt{|y|}}{x^2+y^2} \neq 0.$$

Podemos verlo por redes:

$$\lim_{\substack{x \rightarrow 0 \\ y = mx}} \frac{x\sqrt{|y|}}{x^2+y^2} = \lim_{x \rightarrow 0} \frac{x|x|^{1/2}|m|^{1/2}}{x^2(1+m^2)} = \infty$$

Al no ser diferenciable en $(0,0)$, tampoco es C^1 en ese punto

2.

a) $F(x,y,z) = c$ con $c = F(1,1,0) = -2$.

b)

$$\text{Ic. } z \log(1+z^2) - x^2 - y^2 + 2 = 0$$

• En $x=0, y=0 \rightarrow$ Buscamos $z_0 \in \mathbb{R} /$

$$g(z_0) = z_0 \cdot \log(1+z_0^2) + 2 = 0.$$

Por T^a de Bolzano, $\exists z_0 \in [-2, 0)$ solución de la ecuación. Por T^a de Rolle es única, pues

$$g'(z_0) = \log(1+z_0^2) + \frac{2z_0^2}{1+z_0^2} \neq 0 \quad \forall z_0 \neq 0.$$

Entonces:

$$\begin{array}{l} 1) F(0,0,z_0) + 2 = 0 \\ 2) F \in C^\infty(0,0,z_0) \\ 3) \partial_z F(0,0,z_0) = g'(z_0) \neq 0 \end{array} \left\{ \begin{array}{l} \Rightarrow \exists h \in C^1 / \\ z = h(x,y). \end{array} \right.$$

b) la normal viene dada por $\nabla F(0,0,z_0)$:

$$\nabla F(x,y,z) = (-2x, -2y, g'(z)) \Rightarrow$$

$$\Rightarrow \nabla F(0,0,z_0) = (0,0, g'(z_0)).$$

$$\begin{aligned} \text{Plano tangente: } \{(x,y,z) \in \mathbb{R}^3 : (x,y,z-z_0) \cdot \nabla F(0,0,z_0) = 0\} = \\ = \{(x,y,z) \in \mathbb{R}^3 : z = z_0\} \end{aligned}$$

c) Superficie: $z \cdot \log(1+z^2) - x^2 - y^2 + 2 = 0$

Curva de nivel: $z = C \rightarrow$

$$\rightarrow C \cdot \log(1+C^2) - x^2 - y^2 + 2 = 0$$

Pasa por (1,1): $C \cdot \log(1+C^2) = 0 \Rightarrow C = 0.$

Se trata entonces de la curva $x^2 + y^2 = 2$.

Vector normal en (1,1):

$$g(x,y) = x^2 + y^2 - 2 \rightarrow \nabla g(x,y) = (2x, 2y) \rightarrow \nabla g(1,1) = (2,2).$$

Recta tangente: $(x-1, y-1) \cdot \nabla g(1,1) = 0 \rightarrow$

$$\rightarrow 2(x-1) + 2(y-1) = 0 \rightarrow y = 2-x.$$

3

$$v(y, t) = e^{-ct} u(y - bt, t) \rightarrow \{x = y - bt\}$$

$$\Rightarrow u(x, t) = e^{ct} v(x + bt, t)$$

a)

$$\partial_x u(x, t) = e^{ct} \partial_y v(x + bt, t)$$

$$\partial_t u(x, t) = c \cdot e^{ct} v(x + bt, t) + e^{ct} (\partial_y v(x + bt, t) \cdot b + \partial_t v(x + bt, t))$$

$$b) \partial_{xx}^2 u = e^{ct} \partial_{yy}^2 v$$

Entonces,

$$\partial_t u - a^2 \partial_{xx}^2 u - b \partial_x u - cu =$$

$$= c \cancel{e^{ct} v} + e^{ct} (\cancel{\partial_y v} \cdot b + \partial_t v) - a^2 e^{ct} \partial_{yy}^2 v - b \cancel{e^{ct} \partial_y v} - \cancel{c e^{ct} v}$$

$$\Rightarrow \text{la ecuación queda } \partial_t v = a^2 \partial_{yy}^2 v //$$

4

$$a) \nabla f(x, y) = (y^2 - 2xy, 2xy - x^2) = (0, 0) \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} y(y - 2x) = 0 \\ x(2y - x) = 0 \end{cases} \begin{cases} y=0 \rightarrow x=0 \\ y=2x \rightarrow x=0 \end{cases} \rightarrow (0, 0)$$

$$\partial_x^2 f(x, y) = -2y$$

$$\partial_y^2 f(x, y) = 2x$$

$$\partial_{xy}^2 f(x, y) = 2y - 2x$$

todas nulas en (0, 0).

Vemos directamente que se trata de un punto de silla:

$$f(\varepsilon_x, \varepsilon_y) = \varepsilon_x \varepsilon_y (\varepsilon_y - \varepsilon_x) \Rightarrow \text{tomado } \varepsilon > 0,$$

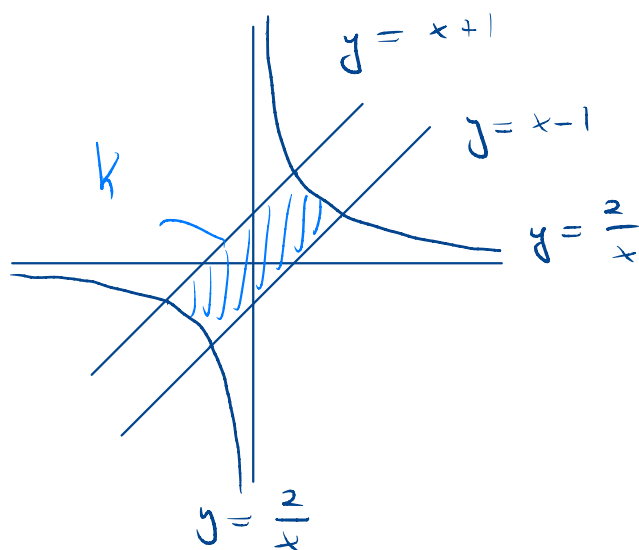
$$f(\varepsilon, 2\varepsilon) = \varepsilon^3 > 0$$

$$f(2\varepsilon, \varepsilon) = -\varepsilon^3 < 0.$$

• No existe los extremos absolutos pues

$$\lim_{\substack{x \rightarrow +\infty \\ y=1}} f(x, y) = -\infty, \quad \lim_{\substack{y \rightarrow +\infty \\ x=1}} f(x, y) = +\infty.$$

b)



1) Puntos críticos: $(0,0)$ (por ∞), ni máx. ni mín.)

2) Extremos condicionados en $xy=2$.

3) " " en $y = x-1$

4) " " en $y = x+1$

1) Vertices: $(2, 1), (1, 2)$
 $(-1, -2), (-2, -1)$

2) $L(x, y, \lambda) = xy^2 - x^2y - \lambda(xy - 2)$

$$\begin{cases} y^2 - 2xy - \lambda y = 0 \\ 2xy - x^2 - \lambda x = 0 \\ xy = 2 \end{cases} \begin{cases} y(y - 2x - \lambda) = 0 \\ x(2y - x - \lambda) = 0 \\ xy = 2 \end{cases} \begin{cases} \textcircled{1} y = 0 \\ \textcircled{2} y - 2x - \lambda = 0 \end{cases}$$

$\textcircled{1} y = 0$
 $x(-x - \lambda) = 0$
 $0 = 2!$
 No tiene sol.

$\textcircled{2} \lambda = y - 2x$
 $x(2y - x - \lambda) = 0$
 $xy = 2$

$\textcircled{2.1} x = 0 \rightarrow$ No tiene solución.

$\textcircled{2.2} \lambda = 2y - x$
 $\lambda = y - 2x$
 $xy = 2$

$$\begin{cases} \lambda = 2y - x \\ \lambda = y - 2x \\ xy = 2 \end{cases} \begin{cases} 2y - x = y - 2x \\ xy = 2 \end{cases} \rightarrow$$

$\lambda = 2y - x$
 $y = -x$
 $xy = 2$

$\rightarrow -x^2 = 2! \rightarrow$ No hay solución.

3) $g(x) = f(x, x-1) = x(x-1)^2 - x^2(x-1) = -x(x-1)$

$g'(x) = -2x + 1 = 0 \Leftrightarrow x = \frac{1}{2} \rightarrow y = -\frac{1}{2} \rightarrow (\frac{1}{2}, -\frac{1}{2})$

$$4) h(x) = f(x, x+1) = x(x+1)$$

$$h'(x) = 2x + 1 = 0 \rightarrow x = -\frac{1}{2} \rightarrow y = \frac{1}{2} \rightarrow \left(-\frac{1}{2}, \frac{1}{2}\right)$$

o Vertices: $(2, 1), (1, 2)$
 $(-1, -2), (-2, -1)$

• Por T^2 Weierstrass, como K es acotado, los extremos absolutos en K existen. Evaluamos:

$$f\left(\frac{1}{2}, -\frac{1}{2}\right) = \frac{-1}{4} \cdot (-1) = \frac{1}{4}, \quad f\left(-\frac{1}{2}, \frac{1}{2}\right) = \frac{-1}{4} \cdot 1 = -\frac{1}{4},$$

$$f(2, 1) = -2, \quad f(1, 2) = 2, \quad f(-1, -2) = -2$$

$$f(-2, -1) = 2 \rightarrow \text{Mín. abs. en } (2, 1), (-1, -2). \\ \text{Máx. abs. en } (-2, -1), (1, 2).$$